

Instanton moduli spaces and bases in coset conformal field theory

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Abstract

Recently proposed relation between conformal field theories in two dimensions and supersymmetric gauge theories in four dimensions predicts the existence of the distinguished basis in the space of local fields in CFT. This basis has a number of remarkable properties, one of them is the complete factorization of the coefficients of the operator product expansion. We consider a particular case of the $U(r)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_p$ which corresponds to a certain coset conformal field theory and describe the properties of this basis. We argue that in the case $p = 2$, $r = 2$ there exist different bases. We give an explicit construction of one of them. For another basis we propose the formula for matrix elements.

1 Introduction

Two-dimensional conformal field theories and $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions were developed independently through years. However, it was observed in the paper by Alday, Gaiotto and Tachikawa [1] that the instanton part of the partition function in $\mathcal{N} = 2$ gauge theory coincides with the conformal block in 2d conformal field theory.

The relation between these two different types of theories is carried out through the intermediate object — moduli space of instantons \mathcal{M} :

$$\begin{array}{ccc}
 & \text{Instanton moduli space } \mathcal{M} & \\
 \swarrow & & \searrow \\
 \text{CFT} & & \mathcal{N} = 2 \text{ gauge theory}
 \end{array} \tag{1.1}$$

The right arrow on this picture symbolises that the path integral for the partition function in $\mathcal{N} = 2$ supersymmetric gauge theory is localized and can be reduced to the integral over the manifold \mathcal{M} (manifold \mathcal{M} is disconnected, its connected components are labeled by some topological characteristics of instantons). The last integral is divergent due to the non-compactness of the manifold \mathcal{M} . However, one can introduce proper regularization in the gauge theory [2] which breaks Lorenzian symmetry, but preserves some of the supersymmetries and makes it possible to apply the localization technique. The regularized integral is localized at the fixed points of an abelian group (torus) which acts on \mathcal{M} by the space-time rotations survived after breaking of Lorenzian symmetry and by the gauge transformation at infinity. The advantage of using of the deformed theory is that the fixed points of the torus are isolated. Hence the partition function is given by the sum of the fixed points contributions. The partition function defined in such a way is usually referred as Nekrasov partition function.

The non-trivial part of (1.1) is represented by the left arrow which means, that there is a natural action of the symmetry algebra \mathcal{A} of some conformal field theory on equivariant cohomologies of \mathcal{M} (see Nakajima's papers [3, 4] for basic examples of such action). Basis in the (localized) equivariant cohomology space can be labeled by the fixed points of the torus [5]. Thus the geometrical construction gives some special basis of states in the highest weight representations $\pi_{\mathcal{A}}$ of the algebra \mathcal{A} . This basis is already remarkable just because of its geometrical origin and possesses many nice properties. Let us list some of them:

- To every torus fixed point $p \in \mathcal{M}$ correspond basic vector $v_p \in \pi_{\mathcal{A}}$. Moreover if $p \in \mathcal{M}_N$, where N is a topological number then the vector v_p has degree N .
- There is a geometrically constructed scalar product on $\pi_{\mathcal{A}}$. Basis v_p is orthogonal under this product and the norm of the vector v_p equals to the determinant of the vector field v in the tangent space of p . The last expression is also denoted by Z_{vec}^{-1} (contribution of the vector multiplet).
- Matrix elements of geometrically defined vertex operators have completely factorized form. The last expressions are also denoted by Z_{bif} (contribution of the bifundamental multiplet).
- There is a commutative algebra (Integrals of Motion) which is diagonalized in the basis v_p . Geometrically this algebra arise from the multiplication on cohomology classes.

Knowledge of the functions Z_{vec} and Z_{bif} allows to compute multi-point conformal blocks on a surface of genus 0 and 1. In CFT they give explicit and remarkably simple expressions for the coefficients of the operator product expansion.

In this paper we consider the particular case of the scheme described above. Namely, we consider the case when \mathcal{M} is the moduli space of $U(r)$ instantons on $\mathbb{C}^2/\mathbb{Z}_p$ where \mathbb{Z}_p acts by formula (z_1 and z_2 are coordinates on \mathbb{C}^2)

$$(z_1, z_2) \mapsto (\omega z_1, \omega^{-1} z_2), \quad \text{where } \omega^p = 1.$$

There are several smooth partial compactifications of this space. One of them can be constructed as follows. Denote by $\mathcal{M}(r, N)$ smooth compactified moduli space of $U(r)$ instantons on \mathbb{C}^2 with topological number N . The set $\mathcal{M}(r, N)^{\mathbb{Z}_p}$ of \mathbb{Z}_p -invariant points on $\mathcal{M}(r, N)$ is a smooth compactification of the space of instantons on $\mathbb{C}^2/\mathbb{Z}_p$. The torus action on $\mathcal{M}(r, N)^{\mathbb{Z}_p}$ induced by the actions on \mathbb{C}^2 and on framing at infinity. The fixed points of this torus are labeled by r -tuples (Y_1, \dots, Y_r) of Young diagrams colored in p colors. Then, there should be a basis labeled by (Y_1, \dots, Y_r) in a representation of some algebra \mathcal{A} .

It was suggested in [6] that the instanton manifold $\mathcal{M} = \bigsqcup_N \mathcal{M}(r, N)^{\mathbb{Z}_p}$ corresponds to the coset conformal field theory

$$\mathcal{A}(r, p) \stackrel{\text{def}}{=} \frac{\widehat{\mathfrak{gl}}(n)_r}{\widehat{\mathfrak{gl}}(n-p)_r}, \quad (1.2)$$

where parameter n is related to equivariant parameters and in general can be arbitrary complex number. Using well known level-rank duality this coset can be rewritten as

$$\mathcal{A}(r, p) = \widehat{\mathfrak{gl}}(p)_r \times \frac{\widehat{\mathfrak{gl}}(n)_r}{\widehat{\mathfrak{gl}}(p)_r \times \widehat{\mathfrak{gl}}(n-p)_r} = \mathcal{H} \times \widehat{\mathfrak{sl}}(p)_r \times \frac{\widehat{\mathfrak{sl}}(r)_p \times \widehat{\mathfrak{sl}}(r)_{n-p}}{\widehat{\mathfrak{sl}}(r)_n}, \quad (1.3)$$

where \mathcal{H} is the Heisenberg algebra. Taking into account the construction of [7] some of these algebras can be rewritten as

	\dots	\dots	\dots	
$p = 3$	$\mathcal{H} \oplus \widehat{\mathfrak{sl}}(3)_1$	\dots	\dots	\dots
$p = 2$	$\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$	$\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \text{NSR}$	\dots	\dots
$p = 1$	\mathcal{H}	$\mathcal{H} \oplus \text{Vir}$	$\mathcal{H} \oplus W_3$	\dots
	$r = 1$	$r = 2$	$r = 3$	

(1.4a)

where Vir is the Virasoro algebra, W_3 is the $\mathfrak{sl}(3)$ W algebra and NSR is the Neveu–Schwarz–Ramond algebra, $N = 1$ superanalogue of the Virasoro algebra. Using the free-field representation of the algebras $\widehat{\mathfrak{sl}}(2)_1$, $\widehat{\mathfrak{sl}}(2)_2$ and $\widehat{\mathfrak{sl}}(3)_1$ and restricting only on some components of \mathcal{M} this table can be rewritten as

	\dots	\dots	\dots	
$p = 3$	$\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$	\dots	\dots	\dots
$p = 2$	$\mathcal{H} \oplus \mathcal{H}$	$\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}$	\dots	\dots
$p = 1$	\mathcal{H}	$\mathcal{H} \oplus \text{Vir}$	$\mathcal{H} \oplus W_3$	\dots
	$r = 1$	$r = 2$	$r = 3$	

(1.4b)

where \mathcal{F} is the Majorana fermion algebra.

In the language of the scheme (1.1) the conjecture of [6] imply that there exists a construction of geometrical action of the algebra (1.2) on equivariant cohomologies of $\mathcal{M} = \bigsqcup_N \mathcal{M}(r, N)^{\mathbb{Z}_p}$. This action was constructed explicitly only in the case of rank one ($r = 1$) in [4]. For higher ranks $r > 1$ a similar construction is not developed so far. However, it can be obtained as a limit of geometrical action of more general algebra constructed by Nakajima in [8]. To be more precise, the author in [8] constructed the action of the so called \mathfrak{gl}_p -toroidal algebra of the level r on equivariant K -theory of the space $\mathcal{M} = \bigsqcup_N \mathcal{M}(r, N)^{\mathbb{Z}_p}$. In some limit equivariant K -theory degenerates to equivariant cohomology and toroidal algebra degenerates to the Vertex operator algebra related to the coset $\mathcal{A}(r, p)$ ¹. The construction based on a limit of toroidal algebra is difficult to accomplish (for $p = 1$ case see [12]). However, using geometrical intuition one can predict the properties of the basis quoted above. It gives the expressions for the conformal blocks which can be compared to the expressions obtained from the standard CFT framework. Below we list main up-to-date achievements in this direction.

¹Algebraic construction of such limit of toroidal algebra is given in the case $r = 1$ [9,10], for $r > 1$ [11]. The geometrical interpretation of the obtained coset algebras is very implicit.

- In the case $p = 1$, $r = 1$ Nakajima [3] defined the geometrical action of the Heisenberg algebra. The fixed points basis corresponds to Jack polynomials, see e.g. [13]. Carlsson and Okounkov gave geometrical construction of the vertex operator in [14].
- The case $p = 1$, $r = 2$ was considered in the paper [1]. The authors conjectured the expression for the multipoint conformal blocks in terms of the Nekrasov instanton partition functions. Alday and Tachikawa in [15] conjectured the existence of the basis which explains these expressions. In [16] explicit algebraic construction of this basis was given.
- The case $p = 1$, $r > 2$ was considered along the lines of [1] by Wyllard [17] (see also [18]). The construction of the basis was done in [19].
- For the case $p = 2$, $r = 2$ V. Belavin and the third author proposed an expression for Whittaker limit of the four-point superconformal block in Neveu-Schwarz sector in terms of Nekrasov instanton partition functions [6]. This result was generalized in [20] for general four-point conformal block. For the results in Ramond sector see [21].
- For $p > 2$. The check of central charges of the coset CFT $\widehat{\mathfrak{sl}}(r)_p \times \widehat{\mathfrak{sl}}(r)_{n-p} / \widehat{\mathfrak{sl}}(r)_n$ from M -theory consideration was performed in [22]. Wyllard [23] considered the Whittaker limit in the case $p = 4$, $r = 2$. Some further checks for this case were made in [24]. In the case of generic p and r some non-trivial checks were done in [23] by use of Kac determinant of the coset CFT.

There exists another compactification of the space of instantons on $\mathbb{C}^2/\mathbb{Z}_p$. Denote by X_p the minimal resolution of the $\mathbb{C}^2/\mathbb{Z}_p$. The moduli space $\mathcal{M}(X_2, r, N)$ of framed torsion free sheaves of rank r on X_p is a smooth compactification of the space of instantons on $\mathbb{C}^2/\mathbb{Z}_p$. The torus action on $\mathcal{M}(X_2, r, N)$ is induced by the torus action on X_p and action on framing at infinity. The fixed points are labelled by p sets of r -tuple of Young diagrams and $p - 1$ vectors $(k_1^i, k_2^i, \dots, k_r^i)$, $1 \leq i \leq p - 1$ of integer numbers. Note that this combinatorial description differs from the description for torus fixed points on $\mathcal{M}(r, N)^{\mathbb{Z}_p}$ in terms of p -colors colored Young diagrams. It is natural to assume that similar algebras act on the equivariant cohomologies of $\mathcal{M}(X_2, r, N)$. In [25, 26] the authors used the space $\mathcal{M}(X_2, 2, N)$ for Nekrasov type expressions of the conformal blocks in the superconformal field theory.

The symmetry algebra for the coset models

$$\frac{\widehat{\mathfrak{sl}}(r)_p \times \widehat{\mathfrak{sl}}(r)_{n-p}}{\widehat{\mathfrak{sl}}(r)_n} \quad (1.5)$$

with generic r and p is not known in explicit form. For example for $r = 2$ and generic p the symmetry algebra is generated by the current $G(z)$ of fractional spin $(p + 4)/(p + 2)$ [27]. This current is non-abelianly braided i.e. the operator product of $G(z)$ with itself contains singularities with incommensurable powers. This fact makes it difficult to study such models. The situation simplifies in three cases: $p = 1$ which corresponds to the Virasoro algebra, $p = 2$ which corresponds to the Neveu-Schwarz-Ramond algebra and $p = 4$ which can be expressed through the abelianly braided model called spin 4/3 parafermionic CFT [28, 29]. For higher ranks the algebraic treatment of the coset model (1.5) becomes even more problematic. Already in the case of $p = 1$ the commutation relations of the corresponding algebra (W_r algebra in this case) are known in explicit terms only for the small ranks. Remarkably, that such obstructions do not appear in geometrical side of the relation (1.1) and the case of generic p and r can be studied in its entirety.

In this paper we continue the study of the case $p = 2$, $r = 2$ as the next example (after $p = 1$ and $r = 2$) where the algebraic treatment is relatively simple². General philosophy suggests the existence of

²Some analysis of the case $p = 4$ and $r = 2$ was done in [23, 24].

the basis in the representation of the algebra $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}$ (see (1.4b)). This basis has geometric origin and gives expressions for the conformal blocks mentioned before. Moreover, the different manifolds $\mathcal{M}(X_2, 2, N)$ and $\mathcal{M}(2, N)^{\mathbb{Z}_2}$ might correspond to different bases.

The appearance of the different bases is a new effect in the case $p > 1$ compared to $p = 1$. Geometrically this is related to the fact that manifolds $\mathcal{M}(X_2, 2, N)$ and $\mathcal{M}(2, N)^{\mathbb{Z}_2}$ are \mathbb{C}^* -diffeomorphic, but not $(\mathbb{C}^*)^2$ -diffeomorphic. Algebraically this leads to the fact that formulae in [6, 20] from the one hand and [25, 26] from the other hand are different. They give the same result because the manifolds $\mathcal{M}(X_2, 2, N)$ and $\mathcal{M}(2, N)^{\mathbb{Z}_2}$ are the compactifications of the same manifold and hence the integrals are equal. In other words these two compactifications give two ways to compute the integral. Equality between results means the nontrivial combinatorial identity.

In section 3 we construct the basis which corresponds to the manifold $\mathcal{M}_2(2, N)$ (to be more precise to its component with $c_1 = 0$). This basis gives [25, 26] expressions for the conformal blocks in the superconformal field theory. As the main tool we use the subalgebra

$$(\mathcal{H} \oplus \text{Vir}) \oplus (\mathcal{H} \oplus \text{Vir}) \subset (\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}).$$

In other words we use an embedding of the direct sum of two algebras for $p = 1$ into the algebra for $p = 2$ (see (1.4b)). Geometrically the appearance of this subalgebra is related to the existence of two points on X_2 invariant under the torus action. Algebraic explanation based on the coset formula

$$\frac{\widehat{\mathfrak{gl}}(n)_r}{\widehat{\mathfrak{gl}}(n-1)_r} \times \frac{\widehat{\mathfrak{gl}}(n-1)_r}{\widehat{\mathfrak{gl}}(n-2)_r} \subset \frac{\widehat{\mathfrak{gl}}(n)_r}{\widehat{\mathfrak{gl}}(n-2)_r}.$$

Using this subalgebras we reduce the basis problem to the $p = 1$ case and use construction of [16].

In section 4 we study the basis corresponding to the manifold $\mathcal{M}(2, N)^{\mathbb{Z}_2}$ (to be more precise only one connected component for each N). We couldn't give an explicit construction of this basis but we conjecture a factorized formula for matrix elements of vertex operators (Z_{bif}) in this basis. We checked this formula comparing two evaluations of the five-point conformal block. In the first case we use the formula mentioned above connected with the hypothetical basis which corresponds to the manifold $\mathcal{M}(2, N)^{\mathbb{Z}_2}$. In the second case we use the basis constructed in section 3. This basis corresponds to the manifold $\mathcal{M}(X_2, 2, N)$.

In the second part of section 4 we study all connected components of $\mathcal{M}(1, N)^{\mathbb{Z}_2}$. In other words it means that we consider the algebra $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$ from the table (1.4a) instead of the algebra $\mathcal{H} \oplus \mathcal{H}$ from the table (1.4b). We will see that there are several classes of connected components labeled by an integer number d and different classes correspond to different bases. The basis constructed in section 3 appears to be a limit when $d \rightarrow \infty$.

The plan of the paper is the following. In section 2 we reproduce all known facts about the basis in the case $p = 1$. The content of the sections 3 and 4 was described above. In 5 we formulate some obvious open questions. In appendix A we discuss the embedding $\text{Vir} \oplus \text{Vir} \subset \mathcal{F} \oplus \text{NSR}$ in more details. In appendices B and C we present some explicit formulae used in sections 3 and 4.

2 The case $p = 1$

In this section we review the construction of the basis in the case $p = 1$ and arbitrary rank r . This example is used to illustrate the general scheme formulated in Introduction. Moreover, some constructions will be used below in section 3.

2.1 Geometrical setup

In this case the geometrical object under consideration is the manifold $\mathcal{M} = \bigsqcup_N \mathcal{M}(r, N)$, where $\mathcal{M}(r, N)$ is the compactified moduli spaces of $U(r)$ instantons on \mathbb{C}^2 with instanton number N (see [30] Ch. 2 or [31] Ch. 3)

$$\mathcal{M}(r, N) \cong \left\{ (B_1, B_2, I, J) \left| \begin{array}{l} \text{(i)} \quad [B_1, B_2] + IJ = 0 \\ \text{(ii)} \quad \text{There is no subspace } S \subsetneq \mathbb{C}^n, \text{ such that} \\ \quad B_\sigma S \subset S \text{ } (\sigma = 1, 2) \text{ and } I_1, \dots, I_r \in S \end{array} \right. \right\} / \text{GL}_N, \quad (2.1)$$

where B_j, I and J are $N \times N, N \times r$ and $r \times N$ complex matrices with the action of GL_N given by

$$g \cdot (B_1, B_2, I, J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}),$$

for $g \in \text{GL}_N$. In (2.1) I_1, \dots, I_r denote the columns of the matrix I . Torus $T = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$ acts on the manifold \mathcal{M} . The action of $(\mathbb{C}^*)^2$ arise from the action of two rotations on \mathbb{C}^2 and $(\mathbb{C}^*)^r$ action arises from the action on framing at infinity. The exact formula reads

$$B_1 \mapsto t_1 B_1; \quad B_2 \mapsto t_2 B_2; \quad I \mapsto It; \quad J \mapsto t_1 t_2 t^{-1} J, \quad (2.2)$$

where $(t_1, t_2, t) \in \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r = T$. Fixed points under the torus action are labeled by the r -tuples of Young diagrams $\vec{Y} = (Y_1, \dots, Y_r)$ and T acts on the tangent space of any fixed point $p_{\vec{Y}} = p_{Y_1, \dots, Y_r}$. For any element $v = (\epsilon_1, \epsilon_2, a) \in \text{Lie}(T)$, where $\epsilon_1, \epsilon_2 \in \mathbb{C}$, a is the diagonal matrix $a = \text{diag}(a_1, \dots, a_r)$ and the determinant of v on the tangent space of $p_{\vec{Y}}$ reads [32, 33]

$$\det v \Big|_{p_{\vec{Y}}} = \prod_{i,j=1}^r \prod_{s \in Y_i} E_{Y_i, Y_j}(a_i - a_j | s) (\epsilon_1 + \epsilon_2 - E_{Y_i, Y_j}(a_i - a_j | s)), \quad (2.3)$$

where

$$E_{Y,W}(x|s) = x - \epsilon_1 l_W(s) + \epsilon_2 (a_Y(s) + 1). \quad (2.3a)$$

In (2.3a) $a_Y(s)$ and $l_W(s)$ are correspondingly the arm length of the box s in the partition Y and the leg length of the box s in the partition W . The inverse of the determinant (2.3) usually called the contribution of the vector hypermultiplet and denoted as

$$Z_{\text{vec}}^{(r)}(\vec{a}, \vec{Y} | \epsilon_1, \epsilon_2) \stackrel{\text{def}}{=} \prod_{i,j=1}^r \prod_{s \in Y_i} \left(E_{Y_i, Y_j}(a_i - a_j | s) (\epsilon_1 + \epsilon_2 - E_{Y_i, Y_j}(a_i - a_j | s)) \right)^{-1}, \quad (2.4)$$

where $\vec{a} = (a_1, \dots, a_r)$. This quantity enters into instanton part of the Nekrasov partition function for pure $U(r)$ gauge theory (without matter)

$$Z_{\text{pure}}^{(r)}(\vec{a}, \epsilon_1, \epsilon_2 | \Lambda) = 1 + \sum_{k=1}^{\infty} \sum_{|\vec{Y}|=k} Z_{\text{vec}}^{(r)}(\vec{a}, \vec{Y} | \epsilon_1, \epsilon_2) \Lambda^{4k}, \quad (2.5)$$

where $\vec{a} = (a_1, \dots, a_r)$ is interpreted as vacuum expectation value of the scalar field and Λ is the scale in gauge theory.

An important quantity is the contribution of the bifundamental matter hypermultiplet [32, 34, 35]. This quantity is defined geometrically and is given by the determinant of the vector field in a fiber of certain bundle over fixed point³ of the torus on $\mathcal{M}(r, N) \times \mathcal{M}(r, N')$

$$Z_{\text{bif}}^{(r)}(m; \vec{a}', \vec{W}; \vec{a}, \vec{Y} | \epsilon_1, \epsilon_2) = \prod_{i,j=1}^r \prod_{s \in Y_i} (\epsilon_1 + \epsilon_2 - E_{Y_i, W_j}(a_i - a'_j | s) - m) \prod_{t \in W_j} (E_{W_j, Y_i}(a'_j - a_i | t) - m), \quad (2.6)$$

³This fixed point is labeled by the pair of r -tuples of Young diagrams \vec{Y} and \vec{W} .

where the parameter m coincides with the mass of bifundamental hypermultiplet. As all the expressions Z_{vec} and Z_{bif} appear to be homogeneous under $a_i \rightarrow \lambda a_i$, $m \rightarrow \lambda m$ and $\epsilon_j \rightarrow \lambda \epsilon_j$ one can fix this freedom by demanding that $\epsilon_1 \epsilon_2 = 1$. We will adopt the notations common in CFT literature

$$\epsilon_1 = b, \quad \epsilon_2 = b^{-1}.$$

Moreover, we assume that $\sum_{j=1}^r a_j = 0$. In particular, below we consider in details the case $r = 1$ and $r = 2$. For $r = 2$ it would be convenient to introduce

$$\mathbb{F}(\alpha|P', \vec{W}; P, \vec{Y}) \stackrel{\text{def}}{=} Z_{\text{bif}}^{(2)}(\alpha; (P', -P'), \vec{W}; (P, -P), \vec{Y}|b, 1/b). \quad (2.7)$$

and

$$\mathbb{N}(P, \vec{Y}) \stackrel{\text{def}}{=} Z_{\text{vec}}^{(2)}((P, -P), \vec{Y}|b, 1/b). \quad (2.8)$$

2.2 Algebraic setup

In this case the conformal field theory under consideration has the symmetry algebra $\mathcal{H} \oplus \mathcal{W}_r$. There is special basis of states in the highest weight representation of this algebra corresponding to the fixed points of the vector field acting on \mathcal{M} . This basis of states diagonalizes an infinite system of commuting quantities (Integrals of Motion) \mathbf{I}_k

$$[\mathbf{I}_k, \mathbf{I}_l] = 0, \quad (2.9)$$

which are elements of the universal enveloping of the algebra $\mathcal{H} \oplus \mathcal{W}_r$. We review the construction of the basis of states in two particular cases $r = 1$ and $r = 2$. For the case of general rank see [19].

2.2.1 Case $r = 1$

Our algebra is Heisenberg algebra with components \mathbf{a}_k and commutation relations⁴

$$[\mathbf{a}_n, \mathbf{a}_m] = n \delta_{n+m,0}. \quad (2.10)$$

The highest weight representation of this algebra (Fock module) is defined by the vacuum state $|0\rangle$

$$\mathbf{a}_n |0\rangle = 0 \quad \text{for } n > 0,$$

and spanned by the vectors of the form

$$\mathbf{a}_{-k_1} \dots \mathbf{a}_{-k_n} |0\rangle, \quad k_1 \geq k_2 \geq \dots \geq k_n. \quad (2.11)$$

One can define another basis

$$|Y\rangle \stackrel{\text{def}}{=} \mathbf{J}_Y^{(1/g)}(x) |0\rangle, \quad (2.12)$$

where $\mathbf{J}_Y^{(1/g)}(x)$ is the Jack polynomial in integral normalization [36] with parameter $g = -b^2$ associated to the partition Y and the following identification is made

$$\mathbf{a}_{-k} = -ib p_k,$$

where p_k are power-sum symmetric polynomials

$$p_k = p_k(x) = \sum_j x_j^k.$$

⁴Here and below we assume that our Heisenberg algebra has no zero mode since it plays artificial role in our construction. In other words we assume that we are considering highest weight representations such that $a_0|0\rangle = 0$.

The basis of states $|Y\rangle$ is usually called Jack basis by transparent reasons. There exists a system of Integrals of Motion \mathbf{I}_k which acts diagonally in Jack basis (2.12). The first two representatives of this family are (here $Q = b + 1/b$)

$$\begin{aligned}\mathbf{I}_1 &= \sum_{k>0} \mathbf{a}_{-k} \mathbf{a}_k, \\ \mathbf{I}_2 &= iQ \sum_{k>0} k \mathbf{a}_{-k} \mathbf{a}_k + \frac{1}{3} \sum_{i+j+k=0} \mathbf{a}_i \mathbf{a}_j \mathbf{a}_k.\end{aligned}\tag{2.13}$$

Another important property of the Jack basis was pointed out in [14]. Namely, consider vertex operator

$$\mathbf{V}_\alpha = e^{(\alpha-Q)\varphi_-(1)} e^{\alpha\varphi_+(1)},\tag{2.14}$$

with $\varphi_+(z) = i \sum_{n>0} \frac{\mathbf{a}_n}{n} z^{-n}$ and $\varphi_-(z) = i \sum_{n<0} \frac{\mathbf{a}_n}{n} z^{-n}$. Define also dual basis $\langle W|$, which is orthogonal to the Jack basis with respect to usual scalar product in the Heisenberg algebra. It was proved in [14] that

$$\langle W|\mathbf{V}_\alpha|Y\rangle = \prod_{s \in Y} \left(b(l_W(s) + 1) - b^{-1} \mathbf{a}_Y(s) - \alpha \right) \prod_{t \in W} \left(b^{-1}(\mathbf{a}_W(t) + 1) - b l_Y(t) - \alpha \right).\tag{2.15}$$

We stress that the Jack basis $|Y\rangle$ is interpreted as the basis of fixed points p_Y of the vector field on instanton manifold \mathcal{M} (in the case of rank one and $\epsilon_1 = b$, $\epsilon_2 = 1/b$) [13]. Integrals of Motion are interpreted as operators of multiplication on cohomology classes. We note that the r.h.s. of (2.15) coincides with (2.6) in the case of $r = 1$, $a = a' = 0$, $m = \alpha$ and $\epsilon_1 = b$, $\epsilon_2 = 1/b$.

$$\langle W|\mathbf{V}_\alpha|Y\rangle = Z_{\text{bif}}^{(1)}(\alpha; 0, W; 0, Y|b, b^{-1})$$

2.2.2 Case $r = 2$

We consider conformal field theory, whose symmetry algebra is $\mathcal{A} = \mathcal{H} \oplus \text{Vir}$ (we use conventions which are specific in this case: there is the factor $1/2$ in commutation relations for a_k generators compared to (2.10))

$$\begin{aligned}[L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\ [a_n, a_m] &= \frac{n}{2}\delta_{n+m,0}, \quad [L_n, a_m] = 0.\end{aligned}\tag{2.16}$$

We will parametrize the central charge c of the Virasoro algebra in a Liouville manner as

$$c = 1 + 6Q^2, \quad \text{where } Q = b + \frac{1}{b}.\tag{2.17}$$

We also need to introduce the operators

$$V_\alpha \stackrel{\text{def}}{=} \mathcal{V}_\alpha \cdot V_\alpha^{\text{vir}},\tag{2.18}$$

where V_α^{vir} is the primary field of the Virasoro algebra with conformal dimension

$$\Delta(\alpha, b) = \alpha(Q - \alpha)\tag{2.19}$$

and \mathcal{V}_α is a free exponential

$$\mathcal{V}_\alpha = e^{2(\alpha-Q)\varphi_-} e^{2\alpha\varphi_+},\tag{2.20}$$

with $\varphi_+(z) = i \sum_{n>0} \frac{a_n}{n} z^{-n}$ and $\varphi_-(z) = i \sum_{n<0} \frac{a_n}{n} z^{-n}$.

Let us consider the highest weight representation of the algebra $\mathcal{H} \oplus \text{Vir}$ parameterized by the momenta P and defined by the vacuum state $|P\rangle$:

$$L_n|P\rangle = a_n|P\rangle = 0, \quad \text{for } n > 0, \quad L_0|P\rangle = \Delta(P)|P\rangle, \quad \langle P|P\rangle = 1.$$

The Virasoro conformal dimension of the state $|P\rangle$ is expressed through the momenta P as

$$\Delta(P) = \frac{Q^2}{4} - P^2.$$

Then the highest weight representation is spanned by the vectors of the form

$$a_{-l_m} \dots a_{-l_1} L_{-k_n} \dots L_{-k_1} |P\rangle, \quad (2.21)$$

$$k = (k_1 \geq k_2 \geq \dots \geq k_n), \quad l = (l_1 \geq l_2 \geq \dots \geq l_m).$$

This representation is irreducible for general values of the momenta P .

In principle, one can choose another basis different from the naive one (2.21). Among the possible bases there is one which is of special interest for us. The defining property of this basis is formulated by the following proposition proved in [16].

Proposition 2.1 There exists unique orthogonal basis $|P\rangle_{\vec{Y}}$ such that

$$\frac{\vec{w} \langle P' | V_\alpha | P \rangle_{\vec{Y}}}{\langle P' | V_\alpha | P \rangle_{\vec{Y}}} = \mathbb{F}(\alpha | P', \vec{W}; P, \vec{Y}). \quad (2.22)$$

In proposition 2.1 we denoted the elements of this basis by $|P\rangle_{\vec{Y}}$ where $\vec{Y} = (Y_1, Y_2)$ stands for the pair of Young diagrams. In (2.22) the function $\mathbb{F}(\alpha | P', \vec{Y}'; P, \vec{Y})$ is defined by (2.6)–(2.7). We note that in geometrical language the basis state $|P\rangle_{\vec{Y}}$ corresponds to the fixed point $p_{\vec{Y}}$ of the vector field. It follows from Proposition 2.1 that the states $|P\rangle_{\vec{Y}}$ form an orthogonal basis

$$\vec{w} \langle P | P \rangle_{\vec{Y}} = \frac{\delta_{\vec{Y}, \vec{W}}}{\mathbb{N}(P, \vec{Y})}, \quad (2.23)$$

where $\delta_{\vec{Y}, \vec{W}} = 0$ if $\vec{Y} \neq \vec{W}$, $\delta_{\vec{Y}, \vec{Y}} = 1$ and function $\mathbb{N}(P, \vec{Y})$ is defined by (2.8).

It will be convenient below to introduce operators $X_{\vec{Y}}(P, b)$:

$$|P\rangle_{\vec{Y}} \stackrel{\text{def}}{=} X_{\vec{Y}}(P, b) |P\rangle, \quad (2.24)$$

and such that $X_{\vec{Y}}(P, b)$ does not contain positive components of \mathcal{A} , i.e.

$$X_{\vec{Y}}(P, b) = \sum_{l+k=|\vec{Y}|} C_{\vec{Y}}^{\vec{l}, \vec{k}}(P, b) a_{-l_m} \dots a_{-l_1} L_{-k_n} \dots L_{-k_1}, \quad (2.25)$$

where $l = \sum l_i$ and $k = \sum k_j$. It can be shown that all the coefficients $C_{\vec{Y}}^{\vec{l}, \vec{k}}(P, b)$ are some polynomials in the momenta P (see examples in [16]).

The system of Integrals of Motion which acts diagonally in the basis $|P\rangle_{\vec{Y}}$ was constructed in [16]. First two representatives of this system are

$$\mathbf{I}_1 = L_0 + 2 \sum_{k>0} a_{-k} a_k, \quad (2.26)$$

$$\mathbf{I}_2 = \sum_{k \neq 0} a_{-k} L_k + 2iQ \sum_{k>0} k a_{-k} a_k + \frac{1}{3} \sum_{i+j+k=0} a_i a_j a_k.$$

This integrable system was studied in [16,37,38]. In particular, it was noticed that the basis of eigenstates is very similar to the Jack basis studied above. The states $|P\rangle_{Y,\emptyset}$ as well as the states $|P\rangle_{\emptyset,Y}$ become the Jack states (2.12) if one expresses the Virasoro generators L_n in terms of bosons. In fact, there are two ways to do it

$$L_n = \sum_{k \neq 0, n} c_k c_{n-k} + i(nQ \mp 2\mathcal{P})c_n, \quad L_0 = \frac{Q^2}{4} - \mathcal{P}^2 + 2 \sum_{k>0} c_{-k} c_k, \quad (2.27)$$

$$[c_n, c_m] = \frac{n}{2} \delta_{n+m,0}, \quad [\mathcal{P}, c_n] = 0, \quad \mathcal{P}|P\rangle = P|P\rangle, \quad \langle P|\mathcal{P} = -P\langle P|.$$

corresponding to the choice of sign in front of operator of the zero mode \mathcal{P} . These two choices define two different sets of bosons c_k , which are related by the unitary transform also called reflection operator [39]. The sign “−” works for the states $|P\rangle_{Y,\emptyset}$ while “+” works for $|P\rangle_{\emptyset,Y}$. For example, taking “−” in (2.27) one can show that

$$|P\rangle_{Y,\emptyset} = \Omega_Y(P) \mathbf{J}_Y^{(1/g)}(x)|P\rangle, \quad (2.28)$$

where $\mathbf{J}_Y^{(1/g)}(x)$ is the Jack polynomial with $g = -b^2$,

$$a_{-k} - c_{-k} = -ib p_k(x),$$

and $\Omega_Y(P)$ is the normalization factor, whose explicit form can be found in [16]. The statement similar to (2.28) is valid for the state $|P\rangle_{\emptyset,Y}$ if one takes the sign “+” in (2.27). At the value $Q = 0$ these two sets of bosons differ by sign and general state $|P\rangle_Y$ can be written as a tensor product of two Jack states [37]. Remarkably, the fact that some of the states become the Jack states after bosonization is valid for any r (see [19]). Using this fact and the “bootstrap” equations suggested in [16,19] one can construct recurrently all basis states.

3 Supersymmetric case ($p = 2, r = 2$)

In this section we construct the basis corresponding to the case $p = 2, r = 2$ from the general scheme. In algebraic side we expect to deal with the algebra $\mathcal{A} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}$.

3.1 Geometrical setup

By X_2 we denote the ALE space, which is the minimal resolution of the factor space $\mathbb{C}^2/\mathbb{Z}_2$. This space can be constructed by gluing two charts \mathbb{C}^2 with coordinates:

$$1: \quad \mathbb{C}^2 (u_1, v_1) \quad u_2 = v_1^{-1}, \quad v_2 = u_1 v_1^2 \quad \quad 2: \quad \mathbb{C}^2 (u_2, v_2) \quad u_1 = u_2^2 v_2, \quad v_1 = u_2^{-1}$$

There is a map $\mathbb{C}^2 \setminus \{0\} \rightarrow X_2$ given in coordinates $u_1 = z_1^2, v_1 = z_2/z_1$ in the first chart and $u_2 = z_1/z_2, v_2 = z_2^2$ in the second chart. Points (z_1, z_2) and $(-z_1, -z_2)$ have the same image under this map. Hence we obtain the projection

$$\pi: X_2 \rightarrow \mathbb{C}^2/\mathbb{Z}_2,$$

which appears to be the minimal resolution of singularity. The preimage of $(0,0) \in \mathbb{C}^2$ is exceptional divisor $C \in X_2$. In the first and the second charts C is given by equations $u_1 = 0$ and $v_2 = 0$ respectively.

The torus action on X_2 arises from the torus action on \mathbb{C}^2 :

$$1: \quad (u_1, v_1) \mapsto (t_1^2 u_1, t_1^{-1} t_2 v_1); \quad 2: \quad (u_2, v_2) \mapsto (t_1 t_2^{-1} u_2, t_2^2 v_2).$$

There are two points which are invariant under the torus action namely p_1 and p_2 origins in the first and second charts respectively.

Let $\mathcal{M} = \bigsqcup_N \mathcal{M}(X_2, 2, N)$ be the moduli space of framed torsion free sheaves on X_2 of rank 2 with Chern classes $c_1 = 0$, $c_2 = N$ [40]. Torus $T = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ acts on the manifold \mathcal{M} . The action of the first $(\mathbb{C}^*)^2$ arise from the action of two rotations on \mathbb{C}^2 and the action of the second $(\mathbb{C}^*)^2$ action arises from the action on framing at infinity.

The points of the torus were described in [41]. They are labeled by the pair of pairs of Young diagrams $\vec{Y}^{(\sigma)} = (Y_1^{(\sigma)}, Y_2^{(\sigma)})$, $\sigma = 1, 2$ and one integer number $k \in \mathbb{Z}$. The pair of Young diagrams $\vec{Y}^{(\sigma)}$ describes the corresponding sheaf $\mathcal{E}_{\vec{Y}^{(\sigma)}, k}$ near the invariant point p_σ and k means that $\mathcal{E}_{\vec{Y}^{(\sigma)}, k}$ is a subsheaf of $\mathcal{O}(kC) + \mathcal{O}(-kC)$.

The determinant of the vector field $v = (\epsilon_1, \epsilon_2, a)$ at the fixed point $p_{\vec{Y}^{(\sigma)}, k}$ equals to [41]

$$\det v \Big|_{p_{\vec{Y}^{(\sigma)}, k}} = \frac{l_{\vec{k}}(\vec{a}|\epsilon_1, \epsilon_2)}{Z_{\text{vec}}^{(2)}(\vec{a} + \epsilon_1 \vec{k}, \vec{Y}^{(1)}|2\epsilon_1, \epsilon_2 - \epsilon_1) Z_{\text{vec}}^{(2)}(\vec{a} + \epsilon_2 \vec{k}, \vec{Y}^{(2)}|\epsilon_1 - \epsilon_2, 2\epsilon_2)}, \quad (3.1)$$

where $\vec{k} = (k, -k)$, function $Z_{\text{vec}}^{(2)}(\vec{a}, \vec{Y}|\epsilon_1, \epsilon_2)$ is given by (2.4) and the factor $l_{\vec{k}}(\vec{a}|\epsilon_1, \epsilon_2)$ is

$$l_{\vec{k}}(\vec{a}|\epsilon_1, \epsilon_2) = (-1)^k \times \begin{cases} l(2a, k)l(\epsilon_1 + \epsilon_2 + 2a, k) & \text{if } k > 0, \\ l(-2a, -k)l(\epsilon_1 + \epsilon_2 - 2a, -k) & \text{if } k < 0, \end{cases} \quad (3.2)$$

where

$$l(x, n) = \prod_{\substack{i, j \geq 1, i+j \leq 2n \\ i+j \equiv 0 \pmod{2}}} (x + (i-1)\epsilon_1 + (j-1)\epsilon_2).$$

Two factors $Z_{\text{vec}}^{(2)}$ in (3.1) arise from the points $p_1, p_2 \in X_2$ invariant under the torus action. The factor $l_{\vec{k}}$ arises from the exceptional divisor. We will call this factor as blow-up factor.

The instanton part of the Nekrasov partition function for the pure $U(2)$ gauge theory on X_2 can be written as [25]

$$Z_{\text{pure}}^{(2, X_2)}(\vec{a}, \epsilon_1, \epsilon_2|\Lambda) = \sum_{k \in \mathbb{Z}} \frac{\Lambda^{2k^2}}{l_{\vec{k}}(\vec{a}|\epsilon_1, \epsilon_2)} Z_{\text{pure}}^{(2)}(\vec{a} + \epsilon_1 \vec{k}, 2\epsilon_1, \epsilon_2 - \epsilon_1|\Lambda) Z_{\text{pure}}^{(2)}(\vec{a} + \epsilon_2 \vec{k}, \epsilon_1 - \epsilon_2, 2\epsilon_2|\Lambda), \quad (3.3)$$

where $Z_{\text{pure}}^{(2)}(\vec{a}, \epsilon_1, \epsilon_2|\Lambda)$ is given by (2.5). Equations (3.1) and (3.3) give some hint about the structure of the basis of states in this case. Namely, the r.h.s. of (3.3) is expressed in terms of two partition functions (corresponding to the case $p = 1$, $r = 2$ from our scheme) with parameters

$$\begin{aligned} \epsilon_1^{(1)} &= 2\epsilon_1, & \epsilon_2^{(1)} &= \epsilon_2 - \epsilon_1, \\ \epsilon_1^{(2)} &= \epsilon_1 - \epsilon_2, & \epsilon_2^{(2)} &= 2\epsilon_2. \end{aligned} \quad (3.4)$$

We note that if we define CFT parameters $b^{(\sigma)}$ by

$$(b^{(\sigma)})^2 = \frac{\epsilon_1^{(\sigma)}}{\epsilon_2^{(\sigma)}},$$

then they are subject to the relation

$$(b^{(1)})^2 + (b^{(2)})^{-2} = -2. \quad (3.5)$$

One can propose that similar relation should hold in the CFT terms too. Namely, in algebraic language we expect that in the algebra $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \mathbf{NSR}$ there are two commuting subalgebras $\mathcal{H} \oplus \mathbf{Vir}$ with the parameters $b^{(1)}$ and $b^{(2)}$ satisfying (3.5). In the next subsection we give explicit construction of these two subalgebras.

3.2 Algebraic setup

As was claimed above this case corresponds to the algebra $\mathcal{A} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}$. Let us first introduce the notations. The commutation relations of the Neveu-Schwarz-Ramond algebra are known to be

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c_{\text{NSR}}}{8}(n^3 - n)\delta_{n+m}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}c_{\text{NSR}}(r^2 - \frac{1}{4})\delta_{r+s,0}, \\ [L_n, G_r] &= \left(\frac{1}{2}n - r\right)G_{n+r}. \end{aligned} \quad (3.6)$$

The central charge c_{NSR} is parameterized as follows

$$c_{\text{NSR}} = 1 + 2Q^2, \quad Q = b + \frac{1}{b}. \quad (3.7)$$

The indexes r and s in (3.6) are either integer (the Ramond sector), or half and odd integer (the Neveu-Schwarz sector). Below we will consider the Neveu-Schwarz sector. The highest weight representation in this case is defined by the vacuum state $|P\rangle_{\text{NS}}$

$$L_n|P\rangle_{\text{NS}} = G_r|P\rangle_{\text{NS}} = 0 \quad \text{for } n, r > 0, \quad L_0|P\rangle_{\text{NS}} = \Delta_{\text{NS}}(Q/2 + P, b)|P\rangle_{\text{NS}}, \quad (3.8)$$

where

$$\Delta_{\text{NS}}(\alpha, b) = \frac{1}{2}\alpha(Q - \alpha). \quad (3.9)$$

3.2.1 Two commutative Virasoro algebras

We will extend our algebra multiplying it by two additional Heisenberg algebras \mathcal{H} and one fermion algebra \mathcal{F} . Let us first multiply the NSR algebra by additional fermion (in the Neveu-Schwarz sector)

$$\{f_r, f_s\} = \delta_{r+s,0}, \quad r, s \in \mathbb{Z} + \frac{1}{2} \quad (3.10)$$

and also we assume that it anticommutes with generators G_r

$$\{G_r, f_s\} = 0. \quad (3.11)$$

It was pointed out in [42–44] that there exists a non-trivial embedding of two commuting Virasoro algebras in $\mathcal{F} \oplus \text{NSR}$ which will be an essential point of our construction⁵. Following [42–44] we can notice that the combinations

$$\begin{aligned} L_n^{(1)} &= \frac{1}{1-b^2}L_n - \frac{1+2b^2}{2(1-b^2)} \sum_{r=-\infty}^{\infty} r : f_{n-r}f_r : + \frac{b}{1-b^2} \sum_{r=-\infty}^{\infty} f_{n-r}G_r, \\ L_n^{(2)} &= \frac{1}{1-b^{-2}}L_n - \frac{1+2b^{-2}}{2(1-b^{-2})} \sum_{r=-\infty}^{\infty} r : f_{n-r}f_r : + \frac{b^{-1}}{1-b^{-2}} \sum_{r=-\infty}^{\infty} f_{n-r}G_r, \end{aligned} \quad (3.12)$$

commute with each other and satisfy the Virasoro commutation relations i.e.

$$\begin{aligned} [L_n^{(1)}, L_m^{(2)}] &= 0, \\ [L_n^{(\sigma)}, L_m^{(\sigma)}] &= (n-m)L_{n+m}^{(\sigma)} + \frac{c^{(\sigma)}}{12}(n^3 - n)\delta_{n+m,0}, \end{aligned} \quad (3.13)$$

⁵The possibility of using of the construction [42–44] in this context was also suggested by Wyllard in [23].

with

$$c^{(\sigma)} = 1 + 6Q^{(\sigma)^2}, \quad Q^{(\sigma)} = b^{(\sigma)} + 1/b^{(\sigma)} \quad \text{and} \quad b^{(1)} = \frac{2b}{\sqrt{2-2b^2}}, \quad (b^{(2)})^{-1} = \frac{2b^{-1}}{\sqrt{2-2b^{-2}}}. \quad (3.13a)$$

We note that the parameters $b^{(1)}$ and $b^{(2)}$ satisfy the relation (3.5).

Consider the highest weight representation $\pi_{\mathcal{F} \oplus \text{NSR}} = \pi_{\mathcal{F}} \otimes \pi_{\text{NSR}}$ of the algebra $\mathcal{F} \oplus \text{NSR}$. In other words we extend the definition of the highest weight vector (3.8) by demanding that

$$f_r |P\rangle_{\text{NS}} = 0, \quad \text{for } r > 0.$$

For general values of the momenta P the highest weight representation $\pi_{\mathcal{F} \oplus \text{NSR}}$ is irreducible. Its character is given by

$$\chi_{\mathcal{F} \oplus \text{NSR}}(q) = \chi_{\mathcal{F}}(q)^2 \chi_{\mathcal{B}}(q), \quad (3.14)$$

where

$$\chi_{\mathcal{F}}(q) = \prod_{k>0} (1 + q^{k-\frac{1}{2}}), \quad \chi_{\mathcal{B}}(q) = \prod_{k>0} \frac{1}{(1 - q^k)}$$

are fermionic and bosonic characters⁶.

We see from (3.12) that there is a natural action of the two Virasoro algebras in the representation $\pi_{\mathcal{F} \oplus \text{NSR}}$. As a representation of $\text{Vir} \oplus \text{Vir}$ it is no longer irreducible and for general values of the momenta P can be decomposed into direct sum of the Verma modules $\pi_{\text{Vir} \oplus \text{Vir}}$ over the algebra $\text{Vir} \oplus \text{Vir}$. The character of any of $\pi_{\text{Vir} \oplus \text{Vir}}$ is given by

$$\chi_{\text{Vir} \oplus \text{Vir}}(q) = \chi_{\mathcal{B}}(q)^2. \quad (3.15)$$

Using the consequence of the Jacobi triple product identity

$$\prod_{k>0} (1 + q^{k-\frac{1}{2}})^2 (1 - q^k) = \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2}} = 1 + 2q^{\frac{1}{2}} + 2q^2 + 2q^{\frac{9}{2}} + \dots$$

we see that

$$\chi_{\mathcal{F} \oplus \text{NSR}}(q) = \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2}} \chi_{\text{Vir} \oplus \text{Vir}}(q), \quad (3.16)$$

which implies the decomposition (see fig. 1)

$$\pi_{\mathcal{F} \oplus \text{NSR}} = \bigoplus_{k \in \mathbb{Z}} \pi_{\text{Vir} \oplus \text{Vir}}^k, \quad (3.17)$$

where $\pi_{\text{Vir} \oplus \text{Vir}}^k$ is the Verma module of $\text{Vir} \oplus \text{Vir}$ with the highest weight $|P, k\rangle$. The highest weight state $|P, k\rangle$ is defined as

$$\begin{aligned} L_n^{(1)} |P, k\rangle &= L_n^{(2)} |P, k\rangle = 0 \quad \text{for } n > 0, \\ L_0^{(1)} |P, k\rangle &= \Delta^{(1)}(P, k) |P, k\rangle, \quad L_0^{(2)} |P, k\rangle = \Delta^{(2)}(P, k) |P, k\rangle, \end{aligned} \quad (3.18)$$

where the conformal dimensions $\Delta^{(1)}(P, k)$ and $\Delta^{(2)}(P, k)$ satisfy the relation

$$\Delta^{(1)}(P, k) + \Delta^{(2)}(P, k) = \Delta_{\text{NS}}(Q/2 + P, b) + \frac{k^2}{2}. \quad (3.19)$$

⁶Usually, the character which is defined as $\text{Tr } q^{L_0} |_{\pi_{\Delta}}$ is proportional to q^{Δ} . We erased these factors for simplicity.

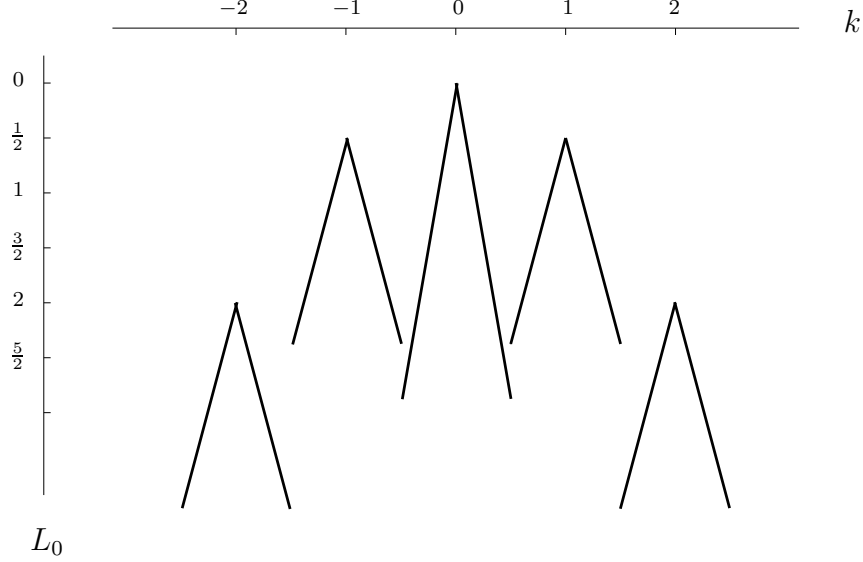


Figure 1: Decomposition of an irreducible representation of the algebra $\mathcal{F} \oplus \mathbf{NSR}$ into direct sum of representations of the algebra $\mathbf{Vir} \oplus \mathbf{Vir}$. Each interior angle corresponds to Verma module $\pi_{\mathbf{Vir} \oplus \mathbf{Vir}}^k$ over the algebra $\mathbf{Vir} \oplus \mathbf{Vir}$ whose conformal dimension is shifted by $k^2/2$ as in (3.19).

Equation (3.19) follows from the relation

$$L_0^{(1)} + L_0^{(2)} = L_0 + L_0^f,$$

where L_0^f is the zeroth component of the stress-energy tensor for the free-fermion

$$L_0^f = \sum_{r=1/2}^{\infty} r f_{-r} f_r.$$

In order to construct the highest weight states $|P, k\rangle$ in more explicit terms and to compute the conformal dimensions $\Delta^{(1)}(P, k)$ and $\Delta^{(2)}(P, k)$ we consider free-field representation for the \mathbf{NSR} algebra. There exist two alternative free-field representations (corresponding to the choice of sign in front of operator \mathcal{P})

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{k \neq 0, n} c_k c_{n-k} + \frac{1}{2} \sum_r (r - \frac{n}{2}) \psi_{n-r} \psi_r + \frac{i}{2} (Qn \mp 2\mathcal{P}) c_n, \\ L_0 &= \sum_{k>0} c_{-k} c_k + \sum_{r>0} r \psi_{-r} \psi_r + \frac{1}{2} \left(\frac{Q^2}{4} - \mathcal{P}^2 \right), \\ G_r &= \sum_{n \neq 0} c_n \psi_{r-n} + i(Qr \mp \mathcal{P}) \psi_r, \quad \mathcal{P}|P\rangle_{\mathbf{NS}} = P|P\rangle_{\mathbf{NS}}, \end{aligned} \tag{3.20}$$

where the operator of zero mode \mathcal{P} , bosonic components c_n and fermionic components ψ_r satisfy commutation relations

$$\begin{aligned} [c_n, c_m] &= n \delta_{n+m, 0}, \quad \{\psi_r, \psi_s\} = \delta_{r+s, 0}, \\ [\mathcal{P}, c_n] &= [\mathcal{P}, \psi_r] = 0. \end{aligned} \tag{3.21}$$

It is convenient to introduce the combinations

$$\chi_r = f_r - i\psi_r,$$

then one can show that the state

$$|P, k\rangle = \Omega_k(P) \chi_{-\frac{1}{2}} \chi_{-\frac{3}{2}} \cdots \chi_{-\frac{2|k|-1}{2}} |\text{vac}\rangle, \quad (3.22)$$

is the highest weight vector, i.e. it satisfies the conditions (3.18) and $|\text{vac}\rangle$ is the vacua state defined by

$$c_n |\text{vac}\rangle = \psi_r |\text{vac}\rangle = f_r |\text{vac}\rangle = 0, \quad \text{for } n, r > 0.$$

Last statement can be derived using the relations

$$\begin{aligned} [L_n^{(1)} + L_n^{(2)}, \chi_r] &= -\left(\frac{n}{2} + r\right) \chi_{r+n}, \\ [bL_n^{(1)} + b^{-1}L_n^{(2)}, \chi_r] &= -((n+r)Q \mp \mathcal{P}) \chi_{r+n} + i \sum_{m \neq 0} c_m \chi_{r+n-m}. \end{aligned} \quad (3.23)$$

The choice of sign in front of the operator of the zero mode \mathcal{P} in (3.20) corresponds to $k > 0$ or $k < 0$ in (3.22). Choosing “ \mp ” in (3.20) we define two different sets of generators c_k and ψ_r . Similarly to the bosonic case they are related by some unitary transform (in particular if $Q = 0$ they just differ by a sign).

Using (3.22) one can compute

$$\Delta^{(1)}(P, k) = \frac{(Q^{(1)})^2}{4} - \left(P^{(1)} + \frac{kb^{(1)}}{2}\right)^2, \quad \Delta^{(2)}(P, k) = \frac{(Q^{(2)})^2}{4} - \left(P^{(2)} + \frac{k}{2b^{(2)}}\right)^2, \quad (3.24)$$

where parameters $b^{(\sigma)}$ and $Q^{(\sigma)}$ are given by (3.13a) and

$$P^{(1)} = \frac{P}{\sqrt{2-2b^2}} \quad \text{and} \quad P^{(2)} = \frac{P}{\sqrt{2-2b^{-2}}}. \quad (3.25)$$

One can also define the state $\langle k', P' |$ conjugated to (3.22)

$$\langle k', P' | = \Omega_{k'}(P') \langle \text{vac} | \chi_{\frac{2|k'|-1}{2}} \cdots \chi_{\frac{1}{2}}. \quad (3.26)$$

This choice is consistent with the following conjugation $f_r^+ = -f_{-r}$. We chose the normalization factors $\Omega_k(P)$ in (3.22) and (3.26) such that

$$|P, k\rangle = \left((G_{-\frac{1}{2}})^{k^2} + \dots\right) |P\rangle, \quad \langle k', P' | = \langle P' | \left((G_{\frac{1}{2}})^{k'^2} + \dots\right), \quad (3.27)$$

where omitted terms have smaller degree in G . One can find that

$$\Omega_k(P) = \frac{1}{2} \prod_{m+n \leq 2|k|} (2P + mb + nb^{-1}). \quad (3.28)$$

This normalization is standard in CFT and from the other side it coincides with geometrical normalization. The norm of the state $|P, k\rangle$ equals to the determinant of the vector field⁷

$$\langle k, P | P, k \rangle = \det v \Big|_{P(\emptyset, \emptyset), (\emptyset, \emptyset), k} \quad (3.29)$$

and coincides with the factor (3.2).

⁷Note that states $|P, k\rangle$ and $\langle k', P' |$ cannot be represented in form (3.18) and (3.26) simultaneously.

3.2.2 Construction of the basis

Now we can multiply our algebra $\mathcal{F} \oplus \text{NSR}$ by two additional Heisenberg algebras $\mathcal{H} \oplus \mathcal{H}$ with generators h_n and w_n

$$[h_n, h_m] = [w_n, w_m] = n\delta_{n+m,0}, \quad [h_n, w_m] = 0. \quad (3.30)$$

The sets of bosons w_n and h_n have different nature. In particular, the bosons w_n are analogous to the bosons \mathbf{a}_n and a_n considered in section 2 and enter into vertex operators in non-symmetric way (see e.g. (3.36)–(3.37) and compare it to (2.14) and (2.20)). Contrary, the bosons h_n always enter in vertex operators in a symmetric way (see (3.38)). From the point of view of scheme (1.4a) the bosons w_n correspond to the factor \mathcal{H} in $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \text{NSR}$, while the bosons h_n belong to the free-field representation for $\widehat{\mathfrak{sl}}(2)_2$ algebra.

We define also another set of generators

$$a_n^{(1)} = \frac{1}{\sqrt{2-2b^2}}(w_n - ibh_n), \quad a_n^{(2)} = \frac{1}{\sqrt{2-2b^{-2}}}(w_n - ib^{-1}h_n), \quad (3.31)$$

such that

$$[a_n^{(\sigma)}, a_m^{(\rho)}] = \frac{n}{2}\delta_{n+m,0}\delta_{\sigma,\rho}, \quad \sigma, \rho = 1, 2. \quad (3.32)$$

Thus in the algebra $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}$ we have two subalgebras $\mathcal{H} \oplus \text{Vir}$ with generators $a_n^{(\sigma)}$ and $L_n^{(\sigma)}$ for $\sigma = 1, 2$ which satisfy (3.13), (3.32) and obvious relations

$$[L_n^{(\sigma)}, a_m^{(\rho)}] = 0.$$

We note that bosons $a_n^{(1)}$ and $a_n^{(2)}$ enter in our construction in a completely symmetric way (together with the symmetry $b \rightarrow 1/b$). For each of these subalgebras we can define integrable system (2.26):

$$\begin{aligned} \mathbf{I}_1^{(\sigma)} &= L_0^{(\sigma)} + 2 \sum_{k>0} a_{-k}^{(\sigma)} a_k^{(\sigma)}, \\ \mathbf{I}_2^{(\sigma)} &= \sum_{k \neq 0} a_{-k}^{(\sigma)} L_k^{(\sigma)} + 2iQ \sum_{k>0} k a_{-k}^{(\sigma)} a_k^{(\sigma)} + \frac{1}{3} \sum_{i+j+k=0} a_i^{(\sigma)} a_j^{(\sigma)} a_k^{(\sigma)}. \end{aligned} \quad (3.33)$$

The eigenvectors for this integrable system can be easily found. At first, we redefine the highest weight states (3.18) by demanding that

$$h_n |P, k\rangle = w_n |P, k\rangle = 0 \quad \text{for } n > 0.$$

Then the eigenvectors can be written in the form

$$|P, k\rangle_{\vec{Y}^{(1)}, \vec{Y}^{(2)}} \stackrel{\text{def}}{=} X_{\vec{Y}^{(1)}} \left(P^{(1)} + \frac{kb^{(1)}}{2}, b^{(1)} \right) X_{\vec{Y}^{(2)}} \left(P^{(2)} + \frac{k}{2b^{(2)}}, b^{(2)} \right) |P, k\rangle, \quad (3.34)$$

where $\vec{Y}^{(1)}$ and $\vec{Y}^{(2)}$ are two pairs of the Young diagrams and parameters $b^{(\sigma)}$ and $P^{(\sigma)}$ are given by (3.13a) and (3.25). Operators $X_{\vec{Y}^{(\sigma)}}(P^{(\sigma)}, b^{(\sigma)})$ in (3.34) are given by (2.24) and consists of generators $L_{-n}^{(\sigma)}$ and $a_{-n}^{(\sigma)}$.

We claim that the basis (3.34) factorizes certain primary operators analogous to (2.18). It is remarkable that compared to the case $p = 1$ we have infinitely many of them

$$\mathbb{V}_\alpha^{(m)} \quad m \in \mathbb{Z}, \quad (3.35)$$

which corresponds to the highest weight states $|P, m\rangle$ due to the operator–state correspondence. Only the field $V_\alpha^{(0)}$ corresponds to the primary field of the NSR algebra, the rest correspond to descendant fields with the conformal dimensions under the “total” stress-energy tensor $T(z) + T^f(z)$

$$\Delta_{\text{NS}}(\alpha) + \frac{m^2}{2},$$

where $T^f(z)$ is the stress-energy tensor for the Majorana fermion f_r . The first few examples of the fields $V_\alpha^{(m)}$ can be easily calculated:

$$\begin{aligned} \mathbb{V}_\alpha^{(0)}(z) &= \Phi_\alpha^{\text{NS}}(z) \cdot \mathcal{W}_\alpha(z), \\ \mathbb{V}_\alpha^{(1)}(z) &= (\alpha f(z) \Phi_\alpha^{\text{NS}}(z) + \Psi_\alpha^{\text{NS}}(z)) e^{i\phi(z)} \mathcal{W}_\alpha(z), \\ \mathbb{V}_\alpha^{(-1)}(z) &= ((Q - \alpha) f(z) \Phi_\alpha^{\text{NS}}(z) + \Psi_\alpha^{\text{NS}}(z)) e^{-i\phi(z)} \mathcal{W}_\alpha(z), \end{aligned} \quad (3.36)$$

where Φ_α^{NS} is the primary field of the NSR algebra with conformal dimension $\Delta(\alpha) = \frac{1}{2}\alpha(Q - \alpha)$, Ψ_α^{NS} its super partner with the dimension $\Delta(\alpha) + 1/2$,

$$f(z) = \sum_r f_r z^{r+1/2}, \quad \phi(z) = i \sum_{n \neq 0} \frac{h_n}{n} z^{-n}$$

and \mathcal{W}_α is a free exponential

$$\mathcal{W}_\alpha = e^{(\alpha-Q)\varphi_-} e^{\alpha\varphi_+}, \quad (3.37)$$

with $\varphi_+ = i \sum_{n>0} \frac{w_n}{n} z^{-n}$ and $\varphi_-(z) = i \sum_{n<0} \frac{w_n}{n} z^{-n}$. For general m the field $\mathbb{V}_\alpha^{(m)}$ has a form

$$\mathbb{V}_\alpha^{(m)} = D^m[\Phi_\alpha^{\text{NS}}(z), f(z)] e^{im\phi(z)} \mathcal{W}_\alpha(z), \quad (3.38)$$

where $D^m[\Phi_\alpha^{\text{NS}}(z), f(z)]$ is some descendant field on a level $m^2/2$.⁸

The commutation relations of the primary fields Φ_α^{NS} , Ψ_α^{NS} and \mathcal{W}_α with generators L_n , a_n , w_n , G_r and f_r can be summarized as

$$\begin{aligned} [L_n, \Phi_\alpha^{\text{NS}}] &= (z^{n+1} \partial_z + (n+1)\Delta(\alpha)z^n) \Phi_\alpha^{\text{NS}}, \\ [L_n, \Psi_\alpha^{\text{NS}}] &= (z^{n+1} \partial_z + (n+1)(\Delta(\alpha) + 1/2)z^n) \Psi_\alpha^{\text{NS}}, \\ [G_r, \Phi_\alpha^{\text{NS}}] &= z^{r+1/2} \Psi_\alpha^{\text{NS}}, \\ \{G_r, \Psi_\alpha^{\text{NS}}\} &= (z^{r+1/2} \partial_z + (2r+1)\Delta(\alpha)z^{r-1/2}) \Phi_\alpha^{\text{NS}}, \\ [w_n, \mathcal{W}_\alpha(z)] &= -i\alpha z^n \mathcal{W}_\alpha, \quad \text{for } n < 0, \\ [w_n, \mathcal{W}_\alpha(z)] &= i(Q - \alpha) z^n \mathcal{W}_\alpha, \quad \text{for } n > 0. \end{aligned} \quad (3.39)$$

Let us consider the matrix elements

$$\mathfrak{F}(\alpha, m | P', k', \vec{W}^{(1)}, \vec{W}^{(2)}; P, k, \vec{Y}^{(1)}, \vec{Y}^{(2)}) \stackrel{\text{def}}{=} \frac{\bar{w}^{(1), \bar{W}^{(2)}} \langle k', P' | \mathbb{V}_\alpha^{(m)} | P, k \rangle_{\vec{Y}^{(1)}, \vec{Y}^{(2)}}}{\langle k', P' | \mathbb{V}_\alpha^{(m)} | P, k \rangle}. \quad (3.40)$$

⁸Geometrical definition of the vertex operator in [14] (for the case of Hilbert schemes) depends on the line bundle on the surface. It is natural to expect that the vertex operator $\mathbb{V}_\alpha^{(m)}$ corresponds to the line bundle $\mathcal{O}(mC)$ on the surface X_2

Proposition 3.1 We propose that

$$\begin{aligned} \mathfrak{F}(\alpha, m|P', k', \vec{W}^{(1)}, \vec{W}^{(2)}; P, k, \vec{Y}^{(1)}, \vec{Y}^{(2)}) &= \mathbb{F}\left(\alpha^{(1)} + \frac{mb^{(1)}}{2}, b^{(1)} \middle| P'_1 + \frac{k'b^{(1)}}{2}, \vec{W}^{(1)}, P_1 + \frac{kb^{(1)}}{2}, \vec{Y}^{(1)}\right) \times \\ &\times \mathbb{F}\left(\alpha^{(2)} + \frac{m}{2b^{(2)}}, b^{(2)} \middle| P'_2 + \frac{k'}{2b^{(2)}}, \vec{W}^{(2)}, P_2 + \frac{k}{2b^{(2)}}, \vec{Y}^{(2)}\right), \end{aligned} \quad (3.41)$$

where

$$\alpha^{(1)} = \frac{\alpha}{\sqrt{2-2b^2}}, \quad \alpha^{(2)} = \frac{\alpha}{\sqrt{2-2b^{-2}}};$$

and parameters b_j and P_j are given by (3.13a) and (3.25) and function \mathbb{F} by (2.6)–(2.7).

We note that Proposition 3.1 suggests the following identification

$$\mathbb{V}_\alpha^{(m)}(z) = V_{\alpha^{(1)}+mb^{(1)}/2}^{(1)}(z) \cdot V_{\alpha^{(2)}+m/2b^{(2)}}^{(2)}(z), \quad (3.42)$$

where by $V_\alpha^{(\sigma)}$ for $\sigma = 1, 2$ we denoted primary operator (2.18) constructed for one of two subalgebras $\mathcal{H} \oplus \text{Vir}$:

$$(\mathcal{H} \oplus \text{Vir})_\sigma \subset \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}.$$

We have checked equality (3.41) by explicit computations on lower levels. For further confirmations see appendix A.

For practical purposes it is also useful to compute the ratio of the matrix elements (blow-up factors)

$$l(\alpha, m|P', k', P, k) \stackrel{\text{def}}{=} \begin{cases} \frac{\langle k', P' | \mathbb{V}_\alpha^{(m)} | P, k \rangle}{\langle P' | \mathbb{V}_\alpha^{(0)} | P \rangle}, & \text{if } k + k' + m = 2n, \\ \frac{\langle k', P' | \mathbb{V}_\alpha^{(m)} | P, k \rangle}{\langle P' | \mathbb{V}_\alpha^{(\pm 1)} | P \rangle}, & \text{if } k + k' + m = 2n + 1. \end{cases} \quad (3.43)$$

Proposition 3.2 The factors (3.43) are given by

$$l(\alpha, m|P', k', P, k) = \begin{cases} \prod_{i,j} s_{\text{even}}\left(\alpha + P'_i + P_j, \frac{m+k'_i+k_j}{2}\right) & \text{if } m + k + k' \text{ is even} \\ \prod_{i,j} s_{\text{odd}}\left(\alpha + P'_i + P_j, \text{int}\left(\frac{m+k'_i+k_j}{2}\right)\right), & \text{if } m + k + k' \text{ is odd} \end{cases} \quad (3.44)$$

where $\vec{P} = (P, -P)$, $\vec{k} = (k, -k)$, $\vec{P}' = (P', -P')$, $\vec{k}' = (k', -k')$ and $\text{int}(x) = \text{sgn}(x)[|x|]$ is the integer part of x and for $n \geq 0$

$$\begin{aligned} s_{\text{even}}(x, n) &= 2^{-\frac{n^2}{2}} \prod_{\substack{i,j \geq 1, i+j \leq 2n \\ i+j \equiv 0 \pmod{2}}} (x + (i-1)b + (j-1)b^{-1}), \\ s_{\text{odd}}(x, n) &= 2^{-\frac{n(n+1)}{2}} \prod_{\substack{i,j \geq 1, i+j \leq 2n+1 \\ i+j \equiv 1 \pmod{2}}} (x + (i-1)b + (j-1)b^{-1}), \end{aligned}$$

while for $n < 0$ we have

$$s_{\text{even}}(x, n) = (-1)^n s_{\text{even}}(Q - x, -n), \quad s_{\text{odd}}(x, n) = s_{\text{odd}}(Q - x, -n).$$

The proof of this proposition can be done by Coulomb integrals method and will be published elsewhere (see also appendix A).

4 Supersymmetric case: another compactification

The basis constructed in section 3 corresponds to the manifold of moduli of framed torsion free sheaves on X_2 . As was mentioned in the Introduction there is another partial compactification of the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_2$. This compactification will be explored in this section.

4.1 Another compactification

Recall that $\mathcal{M}(r, N)$ denotes the compactified moduli spaces of $U(r)$ instantons on \mathbb{C}^2 with the instanton number N . For any numbers $q_1, q_2, \dots, q_r = 0, 1$ there is a natural action of \mathbb{Z}_2 on $\mathcal{M}(r, N)$:

$$B_1 \mapsto -B_1; \quad B_2 = -B_2; \quad I = Iq; \quad J = qJ,$$

where $q = \text{diag}((-1)^{q_1}, \dots, (-1)^{q_r})$. Denote by $\mathcal{M}(r, N)^{\mathbb{Z}_2}$ the \mathbb{Z}_2 invariant part of $\mathcal{M}(r, N)$.

The manifold $\mathcal{M}(r, N)^{\mathbb{Z}_2}$ is smooth but not connected. In order to describe connected components consider the N -dimensional tautological vector bundle \mathcal{V} on $\mathcal{M}(r, N)$. Its fiber at the point $p = (B_1, B_2, I, J)$ coincides with the vector space V obtained from the vectors I_1, \dots, I_r by action of an algebra generated by the operators B_1 and B_2 . If $p \in \mathcal{M}(r, N)^{\mathbb{Z}_2}$ then \mathbb{Z}_2 acts on the fiber of \mathcal{V} at p . Then V can be decomposes $V_+ \oplus V_-$, where V_+ is the trivial representation and V_- is the sign representations of \mathbb{Z}_2 . Two points p, q belong to the same component if the dimensions of V_+ at these points coincide. We denote connected components as $\mathcal{M}(r, d, N)$ where $d = N_+ - N_-$, and N_+, N_- equal to the ranks of the bundles \mathcal{V}_+ and \mathcal{V}_- respectively⁹. It is evident that $d \equiv N \pmod{2}$.

Torus action on $\mathcal{M}(r, N)^{\mathbb{Z}_2}$ is given by formula (2.2). Points $p_{\vec{W}}$ fixed under the torus action are labeled by the r -tuples of Young diagrams $\vec{W} = (W_1, \dots, W_r)$. It is convenient to color these diagrams as follows: the box $s \in W_k$ with coordinates (i, j) is white if $i - j + q_k \equiv 0 \pmod{2}$ and black otherwise. The numbers N_+ and N_- equal to the number of white and black boxes respectively.

The determinant of the vector field $v = (\epsilon_1, \epsilon_2, a)$ at the fixed point $p_{\vec{W}}$ equals to [45, 46]

$$\det v \Big|_{p_{\vec{W}}} = Z_{\text{vec}}^\diamond(\vec{a}, \vec{W} | \epsilon_1, \epsilon_2)^{-1} = \prod_{i,j=1}^2 \prod_{s \in W_i^\diamond} E_{w_i, w_j}(a_i - a_j | s) (\epsilon_1 + \epsilon_2 - E_{w_i, w_j}(a_i - a_j | s)), \quad (4.1)$$

where the superscript \diamond means that the product goes over boxes $s \in W_i$ satisfying

$$a_{w_i}(s) + l_{w_j}(s) + 1 + q_i - q_j \equiv 0 \pmod{2}.$$

In this subsection we consider the $r = 2$ case. Following [6] we choose components $\mathcal{M}(2, 0, N)$ for $(q_1, q_2) = (0, 0)$ and $\mathcal{M}(2, -1, N)$ for $(q_1, q_2) = (1, 1)$ ¹⁰. One can compute the Nekrasov partition function for the pure $U(r)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_2$ using these components:

$$Z_{\text{pure}}^\diamond(\vec{a}, \epsilon_1, \epsilon_2 | \Lambda) = \sum_{k=0}^{\infty} \sum_{\diamond} Z_{\text{vec}}^\diamond(\vec{a}, \vec{W} | \epsilon_1, \epsilon_2) \Lambda^{2k}, \quad (4.2)$$

where the second sum goes over pairs of diagrams \vec{W} with $|W| = k$, $N_+ = N_-$ and with white corners or over pairs of diagrams with $|W| = k$, $N_+ = N_- - 1$ and with black corners. As it was conjectured and checked in [6] this function coincides with Whittaker limit of the four-point conformal block in $\mathcal{N} = 1$ supersymmetric conformal field theory.

⁹ The connectedness of $\mathcal{M}(r, d, N)$ follows from its description in terms of Nakajima quiver varieties.

¹⁰ Such components satisfy the condition $q_1 + q_2 + 2(N_+ - N_-) = 0$ which can be interpreted as the vanishing of the first Chern class [45].

From the other side it was conjectured and checked in [25] that the function $Z_{\text{pure}}^{(2, X_2)}(\vec{a}, \epsilon_1, \epsilon_2 | q)$ defined by (3.3) coincides with the same conformal block as well. Hence, these partition functions equal to each other

$$Z_{\text{pure}}^\diamond(\vec{a}, \epsilon_1, \epsilon_2 | q) = Z_{\text{pure}}^{(2, X_2)}(\vec{a}, \epsilon_1, \epsilon_2 | q). \quad (4.3)$$

Summands on the left hand side are labeled by pairs of colored Young diagrams W_1, W_2 . Summands on the right hand side are labeled by pair of pairs of Young diagrams $\vec{Y}^{(\sigma)} = (Y_1^{(\sigma)}, Y_2^{(\sigma)})$, $\sigma = 1, 2$ and one integer number $k \in \mathbb{Z}$. There exists a bijection between these two types of combinatorial data (see for example [36, Sec 1.1 Ex. 8] or [46]). However the sets of summands on the left hand side and on the right hand side of (4.3) are different (see appendix C). The identity (4.3) is nontrivial, we have the equality of sums of different rational functions.

The formula (4.3) follows from the fact that for $N \in \mathbb{Z}$ manifolds $\mathcal{M}(X_2, 2, N)$ and $\mathcal{M}(2, 0, 2N)$ are the compactifications of the same manifold (moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_2$). Hence the integrals of the equivariant forms should be equal. Similarly for $N \in \mathbb{Z} + \frac{1}{2}$ integrals over $\mathcal{M}(X_2, 2, N)$ and $\mathcal{M}(2, -1, 2N)$ should be equal (see also [40] and [47]).

Geometrical arguments from the Introduction suggest the existence of the basis labeled by pair of colored Young diagrams in the representation of the algebra $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{F} \oplus \text{NSR}$. In notation for this basis we use superscript \diamond : $|P\rangle_{\vec{W}}^\diamond$. Norm of the vector $|P\rangle_{\vec{W}}^\diamond$ should equal to the $Z_{\text{vec}}^\diamond(\vec{a}, \vec{W} | \epsilon_1, \epsilon_2)^{-1}$. The basis $|P\rangle_{\vec{W}}^\diamond$ differs from the basis $|P, k\rangle_{\vec{Y}^1, \vec{Y}^2}$ constructed in Section 3 since sets of summands in (4.3) are different.

Although we do not have an explicit construction of such basis, we suggest the formula for matrix element of the vertex operator $\mathbb{V}_\alpha^{(0)}$ (3.36) in this basis

$$\frac{{}_{\vec{W}}\langle P' | \mathbb{V}_\alpha^{(0)} | P \rangle_{\vec{Y}}^\diamond}{\diamond \langle P' | \mathbb{V}_\alpha^{(0)} | P \rangle^\diamond} = Z_{\text{bif}}^\diamond(\alpha; \vec{P}', \vec{W}; \vec{P}, \vec{Y} | b, b^{-1}), \quad (4.4)$$

where

$$Z_{\text{bif}}^\diamond(m; \vec{a}', \vec{W}; \vec{a}, \vec{Y} | \epsilon_1, \epsilon_2) = \prod_{i,j=1}^r \prod_{\diamond} (\epsilon_1 + \epsilon_2 - E_{Y_i, W_j}(a_i - a'_j | s) - m) \prod_{\diamond} (E_{W_j, Y_i}(a'_j - a_i | t) - m)$$

and the product goes over boxes $s \in Y_i$ and $t \in W_j$ satisfying

$$a_{Y_i}(s) + l_{W_j}(s) + 1 + q_{Y_i} - q_{W_j} \equiv 0 \pmod{2}; \quad a_{W_j}(t) + l_{Y_i}(t) + 1 + q_{W_j} - q_{Y_i} \equiv 0 \pmod{2}.$$

We have checked the formula (4.4) computing the five-point conformal block

$$\langle P' | \mathbb{V}_\alpha^{(0)}(q_1) \mathbb{V}_\alpha^{(0)}(q_1 q_2) \mathbb{V}_\alpha^{(0)}(1) | P \rangle,$$

using two different bases, i.e. comparing in lowest orders in q_1 and q_2 the results obtained with the help of (3.41) and (4.4).

We note that (4.4) can be considered as a system of equations for unknown basis vectors $|P\rangle_{\vec{Y}}^\diamond$. Unfortunately, the solution of this system is not unique. This is closely related to the fact that the vertex operator $\mathbb{V}_\alpha^{(0)}$ does not depend on $\widehat{\mathfrak{sl}}(2)_2$ bosons h_n . Additional constraints could be explicit expressions for matrix elements of operators different from $\mathbb{V}_\alpha^{(0)}$. It is unlikely that the matrix elements of the operators $\mathbb{V}_\alpha^{(m)}$ introduced in section 3 have nice factorized form similar to (4.4) for $m \neq 0$.

Note that if $\epsilon_1 + \epsilon_2 = 0$ (in CFT notations $Q = 0$) the equality (4.3) become trivial. Geometrically it is related to the fact that the manifolds $\mathcal{M}(X_2, 2, N)$ and $\mathcal{M}(2, 0, 2N)$ are \mathbb{C}^* -diffeomorphic, where \mathbb{C}^* acts on \mathbb{C}^2 by formula $(z_1, z_2) \mapsto (wz_1, w^{-1}z_2)$. However these manifolds are not diffeomorphic as $(\mathbb{C}^*)^2$ -manifolds because the determinants at fixed points are different.

4.2 The $r = 1$ case

In this subsection we discuss the phenomena of existence of different bases mentioned above. For simplicity we restrict ourself to the case $r = 1$.

Denote by $\mathcal{M}(X_2, 1, N)$ the moduli space of framed torsion free sheaves on X_2 of rank 1 with Chern classes $c_1 = 0$, $c_2 = N$. Torus fixed points are labeled by pairs of Young diagrams $(Y^{(1)}, Y^{(2)})$, $|Y^{(1)}| + |Y^{(2)}| = N$ and the determinant of the vector field $v = (\epsilon_1, \epsilon_2, a)$ at the fixed point $p_{Y^{(1)}, Y^{(2)}}$ equals to (see [41]):

$$\det v \Big|_{p_{Y^{(1)}, Y^{(2)}}} = Z_{\text{vec}}(Y^{(1)}, Y^{(2)} | \epsilon_1, \epsilon_2)^{-1} = Z_{\text{vec}}(Y^{(1)} | 2\epsilon_1, \epsilon_2 - \epsilon_1)^{-1} Z_{\text{vec}}(Y^{(2)} | \epsilon_1 - \epsilon_2, 2\epsilon_2)^{-1}, \quad (4.5)$$

where Z_{vec} is given in (2.4) and we omit \vec{a} since in $r = 1$ case \vec{a} doesn't appear in formulas. Denote by

$$\mathcal{Z}_N = \sum_{|Y^{(1)}| + |Y^{(2)}| = N} Z_{\text{vec}}(Y^{(1)}, Y^{(2)} | \epsilon_1, \epsilon_2).$$

the coefficient in Nekrasov partition function. The expression \mathcal{Z}_N equals to the integral over moduli space $\mathcal{M}(X_2, 1, N)$. From the general scheme it follows that there should be a basis labeled by $(Y^{(1)}, Y^{(2)})$ in representation of the algebra $\mathcal{H} \oplus \mathcal{H}$ (see (1.4b)). The algebraic construction of this basis is similar to one given in Section 3.

From the colored partition side consider all components $\mathcal{M}(1, d, N)$ (with $q_1 = 0$). The torus fixed points $p_W \in \mathcal{M}(1, d, N)$ are labeled by colored Young diagrams W with $d(W) = d$, $|W| = N$. The determinant of the vector field $v = (\epsilon_1, \epsilon_2, a)$ at the fixed point p_W equals to [45, 46]

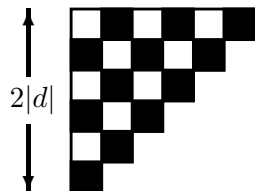
$$\det v \Big|_{p_W} = Z_{\text{vec}}^\diamond(a, \vec{W} | \epsilon_1, \epsilon_2)^{-1} = \prod_{s \in W^\diamond} E_{W,W}(0|s) (\epsilon_1 + \epsilon_2 - E_{W,W}(0|s)), \quad (4.6)$$

where the product goes over boxes $s \in W$ satisfying $a_W(s) + l_W(s) + 1 \equiv 0 \pmod{2}$.

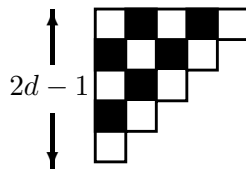
Vectors v_W corresponding to p_W form a basis in representation of the algebra $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$ (see (1.4a)). Combinatorial gradings $d(W)$ and $|W|$ coincide with h_0 grading and principal grading of representation of this algebra. The structure of representation of the algebra $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$ is shown on fig. 2.

Generators e_i from $\widehat{\mathfrak{sl}}(2)_1$ shift d by $+1$, generators f_i by -1 and generators h_i act in subspace with given d . Elements h_i generate the Heisenberg algebra $\mathcal{H} \subset \widehat{\mathfrak{sl}}(2)_1$.

The vectors v_W with given $d(W) = d$ form a basis in representation of the algebra $\mathcal{H} \oplus \mathcal{H}$. It is easy to see that the smallest diagram W_0 with $d(W_0) = d$ consist of $2d^2 - d$ boxes and has a “triangular” form with edge length $2|d|$ for $d \leq 0$ and $2d - 1$ for $d > 0$



for $d < 0$



for $d > 0$

(4.7)

Denote by

$$Z_{d,N} = \sum_{W, d(W)=d, |W|=N} Z_{\text{vec}}^\diamond(W | \epsilon_1, \epsilon_2)$$

the coefficient in the Nekrasov partition function. The expression $Z_{d,N}$ equals to the integral over moduli space $\mathcal{M}(1, d, N)$.

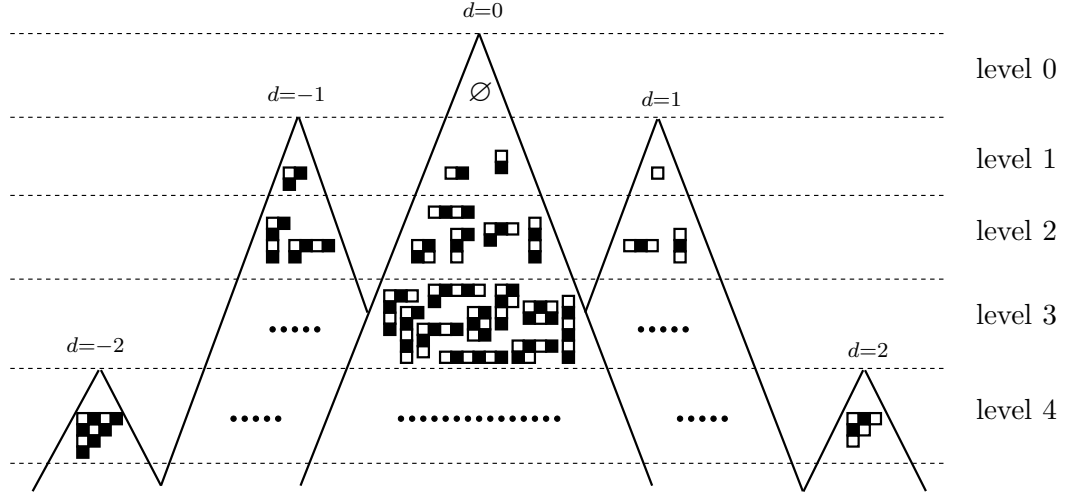


Figure 2: The colored partition basis in the representation of $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$. The interior of each angle corresponds to the representation of $\mathcal{H} \oplus \mathcal{H} \subset \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$ with given value of h_0 . Each colored diagram represents a vector in this representation.

Proposition 4.1 For any integer d

$$Z_{d,2d^2-d+2N} = Z_{0,2N} = \mathcal{Z}_N \quad (4.8)$$

This proposition follows from the fact that the manifolds $\mathcal{M}(1, d, 2d^2 - d + 2N)$ and $\mathcal{M}(X_2, 1, N)$ are birationally isomorphic to the Hilbert scheme of N point on $\mathbb{C}^2/\mathbb{Z}_2$.

The equality (4.8) is an equality of sums. The number of summands from the left hand side and right hand side is the same (this follows from the bijection mentioned above). We will write $\sum \equiv \sum$ if sums are equal and moreover the sets of summands on both sides are the same. Correspondingly we will write $\sum \not\equiv \sum$ if the sums are equal but the sets of summands are different. Direct calculations shows:

$$\begin{aligned} Z_{0,0} &\equiv Z_{1,1} \equiv Z_{-1,3} \equiv Z_{2,6} \equiv Z_{-2,10} \equiv \mathcal{Z}_0. \\ Z_{0,2} &\equiv Z_{1,3} \equiv Z_{-1,5} \equiv Z_{2,8} \equiv Z_{-2,12} \equiv \mathcal{Z}_1. \\ Z_{0,4} &\equiv Z_{1,5} \equiv Z_{-1,7} \equiv Z_{2,10} \equiv Z_{-2,14} \equiv \mathcal{Z}_2. \\ Z_{0,6} &\not\equiv Z_{1,7}, \quad Z_{1,7} \equiv Z_{-1,9} \equiv Z_{2,12} \equiv Z_{-2,16} \equiv \mathcal{Z}_3. \\ Z_{0,8} &\not\equiv Z_{1,9}, \quad Z_{0,8} \not\equiv Z_{-1,11}, \quad Z_{1,9} \not\equiv Z_{-1,11}, \quad Z_{-1,11} \equiv Z_{2,14} \equiv Z_{-2,18} \equiv \mathcal{Z}_4. \\ Z_{0,10} &\not\equiv Z_{1,11}, \quad Z_{0,10} \not\equiv Z_{-1,13}, \quad Z_{1,11} \not\equiv Z_{-1,13}, \\ Z_{0,10} &\not\equiv Z_{2,16}, \quad Z_{1,11} \not\equiv Z_{2,16}, \quad Z_{-1,12} \not\equiv Z_{2,16}, \quad Z_{2,16} \equiv Z_{-2,20} \equiv Z_{3,25} \equiv \mathcal{Z}_5. \end{aligned}$$

These results suggest the following proposition¹¹

- For any d_1, d_2 there exists N such that $Z_{d_1,2d_1^2-d_1+2N} \not\equiv Z_{d_2,2d_2^2-d_2+2N}$
- For any N there exist D such that $Z_{d,2d^2-d+2N} \equiv \mathcal{Z}_N$ for any d , $|d| \geq D$.

¹¹The same phenomena was independently noticed by R. Poghossian [48].

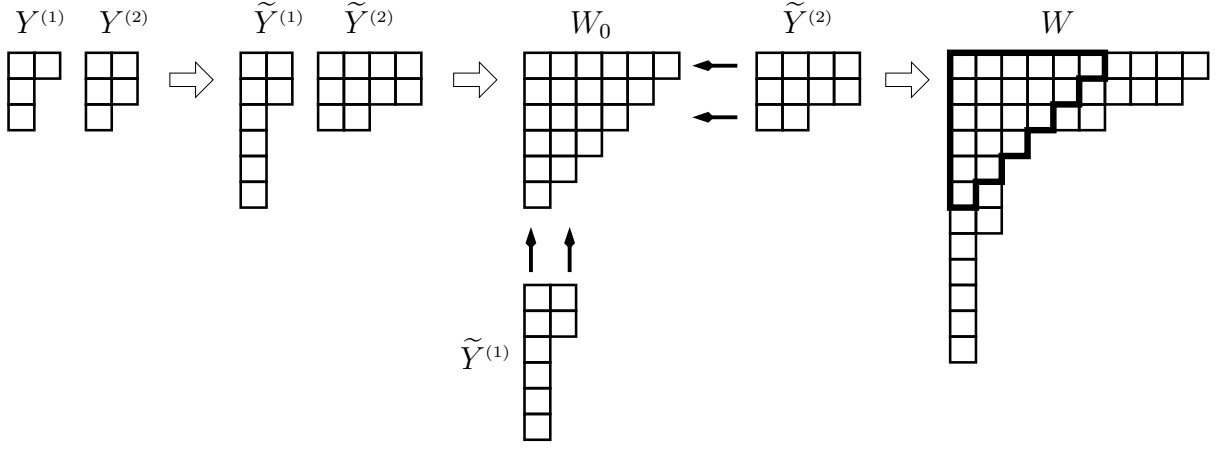


Figure 3: Bijection between the pair $(Y^{(1)}, Y^{(2)})$ and W .

In terms of basis this proposition means that there exists an infinite number of different bases in representation of the algebra $\mathcal{H} \oplus \mathcal{H}$. These bases are numbered by integer number d . Basic vectors in d -th basis are labeled by Young diagrams W with $d(W) = d$. The basis labeled by pairs of Young diagrams $Y^{(1)}, Y^{(2)}$ appears in the limit $d \rightarrow \infty$.

We prove the second assertion:

Proposition 4.2 If $|d| \geq N$, then

$$Z_{d, 2d^2 - d + 2N} \equiv Z_N. \quad (4.9)$$

The proof is based on the explicit bijection: for any pair of Young diagram $Y^{(1)}, Y^{(2)}$ with $|Y^{(1)}| + |Y^{(2)}| = N$ we construct colored Young diagram W with $|W| = 2d^2 - d + 2N$, $d(W) = d$ such that

$$Z_{\text{vec}}^\diamond(W|\epsilon_1, \epsilon_2) = Z_{\text{vec}}(Y^{(1)}, Y^{(2)}|\epsilon_1, \epsilon_2). \quad (4.10)$$

Bijection goes as follows. Denote by W_0 the minimal Young diagram with $d(W_0) = d$. Then $|W_0| = 2d^2 - d$ and W_0 has “triangular” form (4.7). By $\tilde{Y}^{(1)}$ denote diagram obtained from $Y^{(1)}$ by doubling all columns. Similarly, by $\tilde{Y}^{(2)}$ denote diagram obtained from $Y^{(2)}$ by doubling all rows. Then W is obtained by adding diagrams $\tilde{Y}^{(1)}$ and $\tilde{Y}^{(2)}$ to the bottom and to the right of the diagram W_0 respectively (see fig. 3).

The added diagrams $\tilde{Y}^{(1)}$ and $\tilde{Y}^{(2)}$ do not interact since $|d| \geq N$. Then, the identity (4.10) follows from easy combinatorics. \square

In this subsection we considered the $r = 1$ case only. For general r situation is quite similar, there should be a sequence of bases labelled by integer number d . The basis corresponding to $\mathcal{M}_2(r, N)$ appears in the limit $d \rightarrow \infty$.

5 Concluding remarks

1. It would be interesting to give an explicit construction of the basis labeled by colored partitions. As we see in section 4 this basis is not determined by formula for the matrix element (4.4).
2. It would be interesting to generalize results of sections 3 and 4 for the general case $p > 2$. Note that on the instanton moduli side this case is very similar to the $p = 2$ case.

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Appendix A. More on two Virasoro algebras in $\mathcal{F} \oplus \text{NSR}$

In section 3 we observed “strange” relation (3.42) which is equivalent to¹²

$$\begin{aligned} \Phi_{\alpha}^{\text{NS}}(z) &\simeq V_{\alpha^{(1)}}^{\text{Vir}_1}(z) \cdot V_{\alpha^{(2)}}^{\text{Vir}_2}(z), \\ \alpha f(z) \Phi_{\alpha}^{\text{NS}}(z) + \Psi_{\alpha}^{\text{NS}}(z) &\simeq V_{\alpha^{(1)}+b^{(1)}/2}^{\text{Vir}_1}(z) \cdot V_{\alpha^{(2)}+1/2b^{(2)}}^{\text{Vir}_2}(z), \\ (Q - \alpha)f(z) \Phi_{\alpha}^{\text{NS}}(z) + \Psi_{\alpha}^{\text{NS}}(z) &\simeq V_{\alpha^{(1)}-b^{(1)}/2}^{\text{Vir}_1}(z) \cdot V_{\alpha^{(2)}-1/2b^{(2)}}^{\text{Vir}_2}(z), \\ &\dots\dots\dots \end{aligned} \tag{A.1}$$

i.e. for any $m \in \mathbb{Z}$ the product

$$V_{\alpha^{(1)}+mb^{(1)}/2}^{\text{Vir}_1}(z) \cdot V_{\alpha^{(2)}+m/2b^{(2)}}^{\text{Vir}_2}(z)$$

of two primary fields in two CFT’s Vir_1 and Vir_2 with parameters satisfying (3.5) is equal up to normalization to the descendant field on level $m^2/2$ of the field $\Phi_{\alpha}^{\text{NS}}(z)$ in $\mathcal{F} \oplus \text{NSR}$ theory. In operator language this descendant field corresponds to the highest weight vector (3.18). First check which we can perform is to compare conformal dimensions. One can easily find that

$$\Delta(\alpha^{(1)} + mb^{(1)}/2, b^{(1)}) + \Delta(\alpha^{(2)} + m/2b^{(2)}, b^{(2)}) = \Delta_{\text{NS}}(\alpha, b) + \frac{m^2}{2},$$

where $\Delta(\alpha, b)$ and $\Delta_{\text{NS}}(\alpha, b)$ are the conformal dimensions parameterized in Virasoro (Liouville) (2.19) and (3.9) NSR (Super Liouville) manners.

Another more concrete check would be to compare three-point correlation functions. We consider the relation (other relations (A.1) can be treated similarly)

$$\Phi_{\alpha}^{\text{NS}}(z) \simeq V_{\alpha^{(1)}}^{\text{Vir}_1}(z) \cdot V_{\alpha^{(2)}}^{\text{Vir}_2}(z). \tag{A.2}$$

Right hand side of (A.2) is given by the products of two primary operators in two CFT’s Vir_1 and Vir_2 with central charges $c^{(1)}$ and $c^{(2)}$ parameterized as

$$c^{(\sigma)} = 1 + 6 \left(b^{(\sigma)} + \frac{1}{b^{(\sigma)}} \right)^2,$$

¹²For $m = 0$ this relation was noticed in [43].

where $b^{(\sigma)}$ are given by

$$b^{(1)} = \frac{2b}{\sqrt{2-2b^2}}, \quad (b^{(2)})^{-1} = \frac{2b^{-1}}{\sqrt{2-2b^{-2}}}.$$

Let us consider the region $b < 1$. In this case $b^{(1)}$ is real while $b^{(2)}$ is imaginary. For general values of all the parameters we can treat theories \mathbf{Vir}_1 and \mathbf{Vir}_2 as the Liouville field theory [39] with coupling constant $b^{(1)}$ and generalized minimal model [49] (time-like Liouville field theory) with coupling constant $\hat{b}^{(2)}$ (we have fixed the brunch cut as $b^{(2)} = -i\hat{b}^{(2)}$). The three-point functions in both theories

$$\begin{aligned} C(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)} | b^{(1)}) &\stackrel{\text{def}}{=} \langle V_{\alpha_1^{(1)}}(0) V_{\alpha_2^{(1)}}(1) V_{\alpha_3^{(1)}}(\infty) \rangle_{b^{(1)}}, \\ \hat{C}(\hat{\alpha}_1^{(2)}, \hat{\alpha}_2^{(2)}, \hat{\alpha}_3^{(2)} | \hat{b}^{(2)}) &\stackrel{\text{def}}{=} \langle V_{\hat{\alpha}_1^{(2)}}(0) V_{\hat{\alpha}_2^{(2)}}(1) V_{\hat{\alpha}_3^{(2)}}(\infty) \rangle_{\hat{b}^{(2)}}, \end{aligned} \quad (\text{A.3})$$

where

$$\begin{aligned} b^{(1)} &= \frac{2b}{\sqrt{2-2b^2}}, & \alpha^{(1)} &= \frac{\alpha}{\sqrt{2-2b^2}}, \\ (\hat{b}^{(2)})^{-1} &= \frac{2}{\sqrt{2-2b^2}}, & \hat{\alpha}^{(2)} &= \frac{b\alpha}{\sqrt{2-2b^2}}. \end{aligned} \quad (\text{A.4})$$

These three-point functions are known in explicit form [39]

$$C(\alpha_1, \alpha_2, \alpha_3 | b) = \frac{\Upsilon_b(b) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_1 + \alpha_3 - \alpha_2) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1)} \quad (\text{A.5a})$$

and [49]

$$\begin{aligned} \hat{C}(\alpha_1, \alpha_2, \alpha_3 | b) &= \\ &= \frac{\Upsilon_b(b) \Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - b^{-1} + 2b) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3 + b) \Upsilon_b(\alpha_1 + \alpha_3 - \alpha_2 + b) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1 + b)}{\Upsilon_b(2\alpha_1 + b) \Upsilon_b(2\alpha_2 + b) \Upsilon_b(2\alpha_3 + b)}, \end{aligned} \quad (\text{A.5b})$$

where $\Upsilon_b(x)$ is the entire selfdual function (with respect to transformation $b \rightarrow 1/b$), which was defined in [39] by the integral representation

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{b+b^{-1}}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{b+b^{-1}}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right]. \quad (\text{A.6})$$

Equations (A.5) are written up to some factors which can be eliminated by changing of normalization of the primary operators which is always in our hands. For the fields in the left hand side of (A.2) we can define the three-point function

$$C_{\text{NS}}(\alpha_1, \alpha_2, \alpha_3) \stackrel{\text{def}}{=} \langle \Phi_{\alpha_1}^{\text{NSR}}(0) \Phi_{\alpha_2}^{\text{NSR}}(1) \Phi_{\alpha_3}^{\text{NSR}}(\infty) \rangle_b, \quad (\text{A.7})$$

where the average is understood as an average in the Super-Liouville field theory with coupling constant b . Following [50, 51] it has the following explicit form (again up to normalization of primary fields)

$$C_{\text{NS}}(\alpha_1, \alpha_2, \alpha_3) = \frac{\Upsilon_b^{\text{NS}}(2\alpha_1) \Upsilon_b^{\text{NS}}(2\alpha_2) \Upsilon_b^{\text{NS}}(2\alpha_3)}{\Upsilon_b^{\text{NS}}(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b^{\text{NS}}(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b^{\text{NS}}(\alpha_1 + \alpha_3 - \alpha_2) \Upsilon_b^{\text{NS}}(\alpha_2 + \alpha_3 - \alpha_1)}, \quad (\text{A.8})$$

where

$$\Upsilon_b^{\text{NS}}(x) \stackrel{\text{def}}{=} \Upsilon_b\left(\frac{x}{2}\right) \Upsilon_b\left(\frac{x+Q}{2}\right).$$

Using the relation¹³

$$\frac{\Upsilon_{b^{(1)}}(\alpha^{(1)})}{\Upsilon_{\hat{b}^{(2)}}(\hat{\alpha}^{(2)} + \hat{b}^{(2)})} = \frac{\Upsilon_{b^{(1)}}(b^{(1)})}{\Upsilon_{\hat{b}^{(2)}}(\hat{b}^{(2)})\Upsilon_b(b)} b^{\frac{b^2\alpha(Q-\alpha)}{2-2b^2}} \left(\frac{1-b^2}{2}\right)^{\frac{\alpha(Q-\alpha)}{4}-\frac{1}{2}} \Upsilon_b^{\text{NS}}(\alpha), \quad (\text{A.9})$$

one can check that

$$C(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)} | b^{(1)}) \hat{C}(\hat{\alpha}_1^{(2)}, \hat{\alpha}_2^{(2)}, \hat{\alpha}_3^{(2)} | \hat{b}^{(2)}) \simeq C_{\text{NS}}(\alpha_1, \alpha_2, \alpha_3). \quad (\text{A.10})$$

We note that choosing appropriate normalization of the fields one can always set the coefficient of proportionality in (A.10) to be equal to 1.

The ratio of the matrix elements (3.43) can also be interpreted within this framework. Namely, let us assume that $m + k + k'$ is an even number, then

$$l(\alpha, m | P', k', P, k)^2 \simeq \frac{C(\alpha_1^{(1)} + kb^{(1)}/2, \alpha_2^{(1)} + k'b^{(1)}/2, \alpha^{(1)} + mb^{(1)}/2 | b^{(1)})}{C(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha^{(1)} | b^{(1)})} \times \frac{\hat{C}(\hat{\alpha}_1^{(2)} + k/2\hat{b}^{(2)}, \hat{\alpha}_2^{(2)} + k'/2\hat{b}^{(2)}, \hat{\alpha}^{(2)} + m/2\hat{b}^{(2)} | \hat{b}^{(2)})}{\hat{C}(\hat{\alpha}_1^{(2)}, \hat{\alpha}_2^{(2)}, \hat{\alpha}^{(2)} | \hat{b}^{(2)})}, \quad (\text{A.11})$$

where

$$\alpha_1 = \frac{Q}{2} + P, \quad \alpha_2 = \frac{Q}{2} + P',$$

and the sets $(\alpha_1^{(\sigma)}, \alpha_2^{(\sigma)}, \alpha^{(\sigma)})$ and $(\hat{\alpha}_1^{(\sigma)}, \hat{\alpha}_2^{(\sigma)}, \hat{\alpha}^{(\sigma)})$ are related to $(\alpha_1, \alpha_2, \alpha)$ as in (A.4). Equation (A.11) can be checked (again up to normalization of the fields) using the relation

$$\frac{\Upsilon_{b^{(1)}}(\alpha^{(1)})}{\Upsilon_{b^{(1)}}(\alpha^{(1)} + nb^{(1)})} \frac{\Upsilon_{\hat{b}^{(2)}}(\hat{\alpha}^{(2)} + \hat{b}^{(2)} + n/\hat{b}^{(2)})}{\Upsilon_{\hat{b}^{(2)}}(\hat{\alpha}^{(2)} + \hat{b}^{(2)})} = \frac{(-1)^n}{(2-2b^2)^{n^2}} b^{\frac{2bn}{(1-b^2)}(x+nb^{-1}-Q/2)} \times \prod_{\substack{i,j \geq 1, \\ i+j \equiv 0 \pmod{2}}} (\alpha + (i-1)b + (j-1)b^{-1})^2. \quad (\text{A.12})$$

The case when $m + k + k'$ is an odd number can be treated similarly.

Appendix B. Highest weight vectors

In this appendix we give explicit expressions for the highest weight vectors $|P, k\rangle$ defined by (3.18) with

$$\Delta^{(1)}(P, k) = \frac{(Q^{(1)})^2}{4} - \left(P^{(1)} + \frac{kb^{(1)}}{2}\right)^2, \quad \Delta^{(2)}(P, k) = \frac{(Q^{(2)})^2}{4} - \left(P^{(2)} + \frac{k}{2b^{(2)}}\right)^2. \quad (\text{B.1})$$

The state $|P, k\rangle$ belongs to the level $k^2/2$ of the highest weight representation $|P\rangle$ of the algebra $\mathcal{F} \oplus \text{NSR}$. For each value of $k^2/2$ there are exactly two states $|P, k\rangle$ and $|P, -k\rangle$ orthogonal to each other. For

¹³We note that this relation is very similar to the relation used in ref. [52], where the connection between the parafermionic Liouville theory and the three-exponential model [53] was studied.

example, on the level $1/2$ we have

$$\begin{aligned} |P, 1\rangle &= \left(G_{-\frac{1}{2}} + (Q/2 + P)f_{-\frac{1}{2}} \right) |P\rangle_{\text{NS}}, \\ |P, -1\rangle &= \left(G_{-\frac{1}{2}} + (Q/2 - P)f_{-\frac{1}{2}} \right) |P\rangle_{\text{NS}} \end{aligned} \quad (\text{B.2})$$

and on the level 2

$$\begin{aligned} |P, 2\rangle &= \left(G_{-\frac{1}{2}}^4 + (Q/2 + P)^2 G_{-\frac{1}{2}} G_{-\frac{3}{2}} - (Q/2 + P + b)(Q/2 + P + b^{-1}) G_{-\frac{3}{2}} G_{-\frac{1}{2}} - 2(Q + P) G_{-\frac{1}{2}}^3 f_{-\frac{1}{2}} - \right. \\ &\quad \left. - 2(Q/2 + P)^2 (Q + P) G_{-\frac{3}{2}} f_{-\frac{1}{2}} + 2(Q/2 + P + b)(Q/2 + P + b^{-1})(Q + P) G_{-\frac{1}{2}} f_{-\frac{3}{2}} + \right. \\ &\quad \left. + 2(Q/2 + P)(Q/2 + P + b)(Q/2 + P + b^{-1})(Q + P) f_{-\frac{1}{2}} f_{-\frac{3}{2}} \right) |P\rangle_{\text{NS}}, \end{aligned} \quad (\text{B.3a})$$

$$\begin{aligned} |P, -2\rangle &= \left(G_{-\frac{1}{2}}^4 + (Q/2 - P)^2 G_{-\frac{1}{2}} G_{-\frac{3}{2}} - (Q/2 - P + b)(Q/2 - P + b^{-1}) G_{-\frac{3}{2}} G_{-\frac{1}{2}} - 2(Q - P) G_{-\frac{1}{2}}^3 f_{-\frac{1}{2}} - \right. \\ &\quad \left. - 2(Q/2 - P)^2 (Q - P) G_{-\frac{3}{2}} f_{-\frac{1}{2}} + 2(Q/2 - P + b)(Q/2 - P + b^{-1})(Q - P) G_{-\frac{1}{2}} f_{-\frac{3}{2}} + \right. \\ &\quad \left. + 2(Q/2 - P)(Q/2 - P + b)(Q/2 - P + b^{-1})(Q - P) f_{-\frac{1}{2}} f_{-\frac{3}{2}} \right) |P\rangle_{\text{NS}}. \end{aligned} \quad (\text{B.3b})$$

We note that there is an obvious relation

$$|P, k\rangle = |-P, -k\rangle. \quad (\text{B.4})$$

For general values of integer number k one can construct the state $|P, k\rangle$ as described in section 3. Due to (B.4) it is enough to consider only the case $k > 0$. Then we can look for the expression for the vector $|P, k\rangle$ in the form

$$|P, k\rangle = (G_{-\frac{1}{2}}^{k^2} + C_1(P) G_{-\frac{1}{2}}^{k^2-3} G_{-\frac{3}{2}} + \dots) |P\rangle_{\text{NS}}, \quad (\text{B.5})$$

where $(C_1(P) \dots)$ are the coefficients to be determined. As was explained in section 3 the state $|P, k\rangle$ has nice representation in terms of free fields. That means that if we express generators G_r as in (3.20) (for $k > 0$ we have to take the sign “ $-$ ” in (3.20)) and use commutation relations (3.21) we will have

$$|P, k\rangle = \Omega_k(P) \chi_{-\frac{1}{2}} \chi_{-\frac{3}{2}} \dots \chi_{-\frac{2|k|-1}{2}} |\text{vac}\rangle, \quad (\text{B.6})$$

where

$$\chi_r = f_r - i\psi_r.$$

Comparing (B.6) and (B.5) we find all the coefficients $C_j(P)$ unambiguously.

Appendix C. Comparing of $Z_{\text{pure}}^{X_2}$ and $Z_{\text{pure}}^{\diamond}$

We claimed in section 4 that the sets of summands on the left hand side and on the right hand side of the identity (4.3) are different. In this appendix we give an example of such phenomena. The expressions in (4.3) differs first time in coefficient Λ^8 of Λ expansion. For shortness we will use following notation:

$$\epsilon_{i,j} = i\epsilon_1 + j\epsilon_2, \quad a_{i,j} = 2a + i\epsilon_1 + j\epsilon_2.$$

The left hand side of (4.3) can be computed using the formula (4.1) (we omit $\vec{a}, \epsilon_1, \epsilon_2$ in notation). In the order Λ^8 the result reads:

$$\begin{aligned}
& Z_{\text{vec}}^\diamond((4), \emptyset) + Z_{\text{vec}}^\diamond((3, 1), \emptyset) + Z_{\text{vec}}^\diamond(((2, 2), \emptyset) + Z_{\text{vec}}^\diamond(((2, 1, 1), \emptyset) + \\
& Z_{\text{vec}}^\diamond(((1, 1, 1, 1), \emptyset) + Z_{\text{vec}}^\diamond((2, 1), (1)) + Z_{\text{vec}}^\diamond((2), (2)) + Z_{\text{vec}}^\diamond((2), (1, 1)) + \\
& Z_{\text{vec}}^\diamond((1, 1), (2)) + Z_{\text{vec}}^\diamond((1, 1), (1, 1)) + Z_{\text{vec}}^\diamond(((1), (2, 1)) + Z_{\text{vec}}^\diamond(\emptyset(4)) + \\
& Z_{\text{vec}}^\diamond(\emptyset, (3, 1)) + Z_{\text{vec}}^\diamond(\emptyset, (2, 2)) + Z_{\text{vec}}^\diamond(\emptyset, (2, 1, 1)) + Z_{\text{vec}}^\diamond(\emptyset, (1, 1, 1, 1)) = \\
& \frac{1}{\epsilon_1, -3\epsilon_0, 4\epsilon_1, -1\epsilon_0, 2a_{1,1}a_{0,0}a_{1,3}a_{0,2}} + \frac{1}{\epsilon_2, -2\epsilon_{-1}, 3\epsilon_1, -1\epsilon_0, 2a_{1,3}a_{0,2}a_{1,1}a_{0,0}} + \\
& \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_1, -1\epsilon_0, 2a_{2,2}a_{1,1}a_{1,1}a_{0,0}} + \frac{1}{\epsilon_3, -1\epsilon_{-2}, 2\epsilon_2, 0\epsilon_{-1}, 1a_{3,1}a_{2,0}a_{1,1}a_{0,0}} + \\
& \frac{1}{\epsilon_4, 0\epsilon_{-3}, 1\epsilon_2, 0\epsilon_{-1}, 1a_{3,1}a_{2,0}a_{1,1}a_{0,0}} + \frac{1}{a_{2,0}a_{1,-1}a_{1,1}a_{-0,0}a_{1,1}a_{0,0}a_{-1,1}a_{0,2}} + \\
& \frac{1}{\epsilon_1, -1\epsilon_0, 2\epsilon_1, -1\epsilon_0, 2a_{1,-1}a_{0,-2}a_{-1,1}a_{0,2}} + \frac{1}{\epsilon_1, -1\epsilon_0, 2\epsilon_2, 0\epsilon_{-1}, 1a_{1,1}a_{0,0}a_{1,1}a_{0,0}} + \\
& \frac{1}{\epsilon_1, -1\epsilon_0, 2\epsilon_2, 0\epsilon_{-1}, 1a_{1,1}a_{0,0}a_{1,1}a_{0,0}} + \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_2, 0\epsilon_{-1}, 1a_{2,0}a_{1,-1}a_{-2,0}a_{-1,1}} + \\
& \frac{1}{a_{1,-1}a_{0,2}a_{-2,0}a_{-1,1}a_{-1,1}a_{0,0}a_{-1,-1}a_{0,0}} + \frac{1}{\epsilon_1, -3\epsilon_0, 4\epsilon_1, -1\epsilon_0, 2a_{-1,-1}a_{0,0}a_{-1,-3}a_{0,-2}} + \\
& \frac{1}{\epsilon_2, -2\epsilon_{-1}, 3\epsilon_1, -1\epsilon_0, 2a_{-1,-3}a_{0,-2}a_{-1,-1}a_{0,0}} + \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_1, -1\epsilon_0, 2a_{-2,-2}a_{-1,-1}a_{-1,-1}a_{0,0}} + \\
& \frac{1}{\epsilon_3, -1\epsilon_{-2}, 2\epsilon_2, 0\epsilon_{-1}, 1a_{-3,-1}a_{-2,0}a_{-1,-1}a_{0,0}} + \frac{1}{\epsilon_4, 0\epsilon_{-3}, 1\epsilon_2, 0\epsilon_{-1}, 1a_{-3,-1}a_{-2,0}a_{-1,-1}a_{0,0}} = \\
& \frac{16a^4 - 52a^2\epsilon_1^2 + 36\epsilon_1^4 - 92a^2\epsilon_1\epsilon_2 + 177\epsilon_1^3\epsilon_2 - 52a^2\epsilon_2^2 + 294\epsilon_1^2\epsilon_2^2 + 177\epsilon_1\epsilon_2^3 + 36\epsilon_2^4}{2\epsilon_1\epsilon_2a_{-1,-1}a_{1,1}a_{-2,-2}a_{2,2}a_{-3,-1}a_{-1,-3}a_{1,3}a_{3,1}}.
\end{aligned}$$

The right hand side of (4.3) can be computed using the formula (3.1)

$$\begin{aligned}
& Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{\emptyset, \emptyset\}, -2) + Z_{\text{vec}}(\{(2), \emptyset\}, \{\emptyset, \emptyset\}, 0) + Z_{\text{vec}}(\{(1, 1), \emptyset\}, \{\emptyset, \emptyset\}, 0) + \\
& Z_{\text{vec}}(\{\{1\}, \{1\}\}, \{\emptyset, \emptyset\}, 0) + Z_{\text{vec}}(\{\emptyset, (2)\}, \{\emptyset, \emptyset\}, 0) + Z_{\text{vec}}(\{\emptyset, (1, 1)\}, \{\emptyset, \emptyset\}, 0) + \\
& Z_{\text{vec}}(\{(1), \emptyset\}, \{(1), \emptyset\}, 0) + Z_{\text{vec}}(\{(1), \emptyset\}, \{\emptyset, (1)\}, 0) + Z_{\text{vec}}(\{\emptyset, (1)\}, \{(1), \emptyset\}, 0) + \\
& Z_{\text{vec}}(\{\emptyset, (1)\}, \{\emptyset, (1)\}, 0) + Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{(2), \emptyset\}, 0) + Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{(1, 1), \emptyset\}, 0) + \\
& Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{(1), (1)\}, 0) + Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{\emptyset, (2)\}, 0) + Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{\emptyset, (1, 1)\}, 0) + \\
& Z_{\text{vec}}(\{\emptyset, \emptyset\}, \{\emptyset, \emptyset\}, 2) = \\
& \frac{1}{a_{0,0}a_{-2,0}a_{0,-2}a_{-1,-1}a_{-1,-1}, -3a_{-3,-1}a_{-2,-2}} + \frac{1}{\epsilon_3, -1\epsilon_{-2}, 2\epsilon_2, 0\epsilon_{-1}, 1a_{1,1}a_{0,0}a_{0,2}a_{-1,1}} + \\
& \frac{1}{\epsilon_4, 0\epsilon_{-3}, 1\epsilon_2, 0\epsilon_{-1}, 1a_{3,1}a_{2,0}a_{1,1}a_{0,0}} + \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_2, 0\epsilon_{-1}, 1a_{2,0}a_{1,-1}a_{-2,0}a_{-1,1}} + \\
& \frac{1}{\epsilon_3, -1\epsilon_{-2}, 2\epsilon_2, 0\epsilon_{-1}, 1a_{-1,-1}a_{0,0}a_{0,-2}a_{1,-1}} + \frac{1}{\epsilon_4, 0\epsilon_{-3}, 1\epsilon_2, 0\epsilon_{-1}, 1a_{-3,-1}a_{-2,0}a_{-1,-1}a_{0,0}} + \\
& \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_1, -1\epsilon_0, 2a_{1,1}a_{0,0}a_{1,1}a_{0,0}} + \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_1, -1\epsilon_0, 2a_{1,1}a_{0,0}a_{-1,-1}a_{0,0}} + \\
& \frac{1}{\epsilon_{-1}, 1\epsilon_0, 2\epsilon_2, 0\epsilon_{-1}, 1a_{1,1}a_{0,0}a_{-1,-1}a_{0,0}} + \frac{1}{\epsilon_2, 0\epsilon_{-1}, 1\epsilon_1, -1\epsilon_0, 2a_{-1,-1}a_{0,0}a_{-1,-1}a_{0,0}} +
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{\epsilon_1, -3\epsilon_0, 4\epsilon_1, -1\epsilon_0, 2a_1, 1a_0, 0a_1, 3a_0, 2} + \frac{1}{\epsilon_2, -2\epsilon_{-1}, 3\epsilon_1, -1\epsilon_0, 2a_2, 0a_1, -1a_1, 1a_0, 0} + \\
& \frac{1}{\epsilon_1, -1\epsilon_0, 2\epsilon_1, -1\epsilon_0, 2a_1, -1a_0, -2a_{-1}, 1a_0, 2} + \frac{1}{\epsilon_1, -3\epsilon_0, 4\epsilon_1, -1\epsilon_0, 2a_{-1}, -1a_0, 0a_{-1}, -3a_0, -2} + \\
& \frac{1}{\epsilon_2, -2\epsilon_{-1}, 3\epsilon_1, -1\epsilon_0, 2a_{-2}, 0a_{-1}, 1a_{-1}, -1a_0, 0} + \frac{1}{a_0, 0a_2, 0a_0, 2a_1, 1a_1, 3a_3, 1a_2, 2} = \\
& \frac{16a^4 - 52a^2\epsilon_1^2 + 36\epsilon_1^4 - 92a^2\epsilon_1\epsilon_2 + 177\epsilon_1^3\epsilon_2 - 52a^2\epsilon_2^2 + 294\epsilon_1^2\epsilon_2^2 + 177\epsilon_1\epsilon_2^3 + 36\epsilon_2^4}{2\epsilon_1\epsilon_2a_{-1}, -1a_1, 1a_{-2}, -2a_2, 2a_{-3}, -1a_{-1}, -3a_1, 3a_3, 1}.
\end{aligned}$$

We see that results are the same but the sets of summands are different. For example there are only two summands which have degree 8 in variable a , but these summands are different.

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